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Applications of contingent claims theory to microeconomic problems

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Applications of contingent claims theory to microeconomic problems

by

David A. Hennessy

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For the Graduate College

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Dedication

To my Parents

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GENERAL ABSTRACT

In this thesis contingent claims techniques have been applied to various specifications of the economic problem of optimizing the expected value of a welfare function. In paper I we consider the relationship between financial market completeness, corn production, and the corn target price program. Using the observation that the program is similar to a government issued put option, we found that the per acre program benefit, at around $20/acre was quite large, that the program encourages producers to trade options, and that the existence of contingent markets facilitates the policy maker in decoupling agricultural support. In paper II we proposed a method for estimating the expected cost to the government of the corn target price program. The model allows the government to understand the implications for output and budget control of different program parameter choices. This model may be adapted to other economic problems, such as the effects of wage or rent control laws on production and factor use. In paper III we suggest that there is an inconsistency between the structure of existing contingent claims markets and how economists would seem to prefer to approximate demand functions. We propose an alternative structure that is consistent with the preferred approach to demand function approximation, and with the moment based foundations of statistics. In the final paper we propose an alternative perspective on problems involving the maximization of the expected value of a welfare function. We reformulate the objective function in

terms of options. We then show that existing techniques from economics, statistics, and finance theory may be applied to better understand the economic effects of uncertainty. Three standard economic problems are considered; valuation of a risky investment, production under price uncertainty, and the effects of price uncertainty on expected profit.

GENERAL INTRODUCTION

Explanation of Dissertation Format

The papers in this thesis, though similar in subject matter, are entirely self-contained. They each have their own introduction, literature review, conclusions, and references. The papers are linked both through the issues addressed, and through the techniques used. Each paper seeks to understand the effect of nonlinearities on economic decisions and welfare when price is uncertain. All papers apply the financial theory of option pricing to models of economic phenomena. Following the papers is a general summary.

Participant Value and Production Effects of Target Price Programs: A Contingent Claims Approach

It has been noted that the U.S. target price program can be considered as contingent claims issued by the U.S. government to participating farmers. This paper extends the analogy by modeling the acreage requirement as the cost of participation. The results have implications for extension advice regarding participation decisions. It is also shown that the value of the program to producers depends on the existence and accessibility of contingent claims markets, and that the deficiency payments scheme makes it optimal to participate in options markets. Inferences are also made concerning the production effects of alternate forms of income support.

Government Costs of Target Price Supports

The U.S commodity target price program can be considered as the issuance by the U.S. government of contingent claims to participating farmers. This paper models the acreage set-aside requirement as the premium paid for these contingent claims. A rational expectations model of the interrelationships between program parameters, production, futures price, program cost, and producer benefit is developed. The model can accommodate stochasticity on either supply and\or demand. We use the model to estimate the expected government cost of the corn target price program in 1993. The principal innovation in this paper is the accommodation of choice variables in the stochastic model. The approach has applications in many other areas of economics.

Polynomial Price Contracts

This paper compares the approach to functional approximation used in mathematics, statistics, and econometrics with that used in creating contingent financial markets. Evidence is presented that suggests options markets are not optimal. An alternative market structure is proposed that would increase hedging effectiveness, and the risk return tradeoff for hedgers and speculators, respectively. Fewer derivative markets would be required per underlying asset than with options markets. The settlement price of these alternative markets would be some power of the closing futures price. The purpose of this paper is

to advocate that further research is warranted concerning the optimal structure of contingent markets when markets are costly to maintain.

An Alternative Perspective on the Expected Value of a Function: Economic Applications

By approximating the expected value of a function, nonlinear in a stochastic variable, as the sum of values of a sequence of options, we gain additional insights about economic behavior under uncertainty. This is because the respecified behavioral equations contain probabilities and conditional expectations that respond in a predictable manner to changes in the probability distribution. The procedure is formally developed in the context of expected utility maximization when output price is stochastic. It is applied to three problems: to value a risky investment, to study production under price uncertainty, and to study the effect of price uncertainty on expected output when output can be modified in response to realized price.

PAPER I

PARTICIPANT VALUE AND PRODUCTION EFFECTS OF TARGET PRICE PROGRAMS: A CONTINGENT CLAIMS APPROACH

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PARTICIPANT VALUE AND PRODUCTION EFFECTS OF TARGET PRICE PROGRAMS: A CONTINGENT CLAIMS APPROACH

by

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ABSTRACT

It has been noted that the U.S. target price program can be considered as contingent claims issued by the U.S. government to participating farmers. This paper extends the analogy by modeling the acreage requirement as the cost of participation. The results have implications for extension advice regarding participation decisions. It is also shown that the value of the program to producers depends on the existence and accessibility of contingent claims markets, and that the deficiency payments scheme makes it optimal to participate in options markets.

1. INTRODUCTION

The 1990 Farm Bill introduced some important changes into U.S. grain policies. For the first time, producers received deficiency payments based on historical (not actual) yields and acreages; also, they were given the opportunity to plant 15 percent of their program acreage to nonprogram crops. The effect of these changes was to ensure that acreage and input decisions were made in response to expected market conditions. The removal of the link between actual production and government payments also imbued the program with option-like characteristics. As is true for owners of put options, program participants receive the difference (if positive) between a fixed target (or strike) price, and the market price, for a specified volume of production that is independent of actual production. This option-like quality has several interrelated implications, each of which we examine here.

By modeling the acreage reduction requirement as an option premium, one can develop participation criteria that do not depend on personal preferences. Because the participation decision is independent of the risk preferences of producers, we can develop objective measures of the benefits to producers of program participation under different program parameters, market conditions, and land-quality distributions. The procedure used to examine the participation decision differs from that currently used by extension agents and should, therefore, be of direct use to producers.

Because producers can choose to "sell" their government-provided put option on commodity options markets, the program does not have output-enhancing or risk-reducing effects. This raises the question as to the value of deficiency payments schemes when options and futures markets are absent (for example, agricultural commodities in the EC). The third section of the paper compares the certainty equivalent returns for typical midwestern grain producers under different institutional environments. These values indicate that, for producers enrolled in the program, there are benefits from having access to both futures *and* options markets. This result contrasts with that obtained by Lapan et al. (1991) who show that in the absence of government programs, producers have no incentive to participate in options markets. The CER values also show that the benefits of the program, when no futures or options market exists, can be quite large. This is true because a free put option is more valuable if put options cannot be purchased. This latter result means that "decoupled" deficiency payment schemes would have output expansionary effects if they are introduced into market environments where no commodity markets exist. One implication of this argument is that should the EC introduce a farm program that is similar in every respect to that currently used in the United States, the program would not be decoupled.

The last section of the paper presents two theorems about the output expansionary effect of these programs in the presence and absence of contingent claims markets.

2. PREVIOUS WORK

Gardner (1977) originally pointed out the close similarity between the deficiency payments schemes and the yet-to-be-developed commodity options contracts. Gardner (1977) and Kahl (1986) suggested the deficiency payments scheme be replaced with free options. This idea proved impractical until payments were decoupled from actual production in the 1990 Farm Bill. In 1993, the government launched a pilot options program in nine midwestern counties. This program allows producers to choose between program participation or free put options.

Turvey et al. (1988) showed how producers could use options markets to make the program participation decision. Their approach did not, however, allow for either market incompleteness or heterogeneity of land quality, important determinants of the cost of participation. Love and Foster (1990) and Brooks et al. (1992) model the effects of heterogeneous land quality on the participation decision, but do not follow through to decide when a given producer should participate. Innes and Rausser (1989) show how important the assumption of complete markets is in determining the relative effectiveness of different agricultural policies. It is shown that with incomplete markets government price guarantees, even with a set-aside requirement, can improve the welfare of farmers and consumers while also generating enough tax revenue to fund the program. In two subsequent (1990 a, 1990 b) papers Innes identified the conditions under which a target price policy increases welfare in an incomplete

market environment.

There has been considerable analysis conducted on the implications for supply of changing government program parameters. These include Floyd (1965), Lidman and Bawden (1974), Evans (1980), Lee and Helmberger (1985), Miranda and Helmberger (1988), and Perry et al. (1989). Of these studies only one, Lee and Helmberger, attempted to model supply response under a free market regime. This is because supply control of one form or another has been in effect in the United States for most major crops for all but a handful of years since WW II. Gardner (1987) considers some of the political determinants of the magnitude and nature of agricultural support. Chavas et al. (1983) investigates the acreage response to program parameters and futures prices. Marcus and Modest (1986) investigate the cost to the government of the price floor.

To the best of our knowledge, there has been no rigorous attempt to link the program participation decision to the benefits (option value) and costs (option premium or set-aside requirement). This is surprising because as has become obvious since the implementation of the pilot options program, and as we will show in the next section, the participation decision is simply one of determining whether the program parameters are those of a fairly priced option.

3. THE CONTINGENT CLAIMS MODEL

Under the provisions of the 1990 Farm Act a producer eligible for deficiency payments has contracted to idle a certain fraction, α, of base acres (eligible acres), B, in return for a price floor guarantee, P_G, on a pre-specified eligible per acre output, y_0, for each of $(1 - 0.15 - \alpha)B$ acres. The 15 percent of base acres on which no deficiency payment can be made may be planted under any crop except fruits and vegetables. Assume the accessibility of put markets and denote the present value of a put option with strike price P_G as $W(P_G)$. Because the producer can sell puts with strike price P_G to restore the position that would have pertained were there no program, the program benefits are

$$(0.85 - \alpha)\, B\, y_0\, W(P_G) \tag{1}$$

in cash equivalents. Assume (as Black (1976) does) that there is no basis risk, that prices are lognormally distributed, that the short-term interest rate, r, is constant, that there are no transaction costs, borrowing constraints, or restrictions on short sales. Then the benefit to the producer of the program is

$$(0.85 - \alpha)\, B\, y_0\, e^{-r(T-t)}[P_G\, N(-d_2) - F_t\, N(-d_1)], \tag{2}$$

where T is time index value at harvest

t is time index value at sign up for the program

$d_1 = [\ln(F_t/P_G) + 0.5\, \sigma^2\, (T - t)]/\sigma\, (T - t)$

$d_2 = [\ln(F_t/P_G) - 0.5\, \sigma^2\, (T - t)]/\sigma\, (T - t)$

σ is the implied volatility or the annualized standard deviation of the instantaneous rate of return.

F_t is futures price at time t (see Rubinstein 1976 or Myers and Hanson 1993).

The cost is the present value of foregone profits from set-aside land. The idled land will be the 100α percent of lowest quality land. Define PV(.) as the present value operator, $\pi(v)$ as the per acre profit from land of quality v, (v is indexed such that $0 \leq v \leq 1$), J(v) as the cumulative density function of a producer's land quality, and j(v) as the probability density function of v. Then the cost is

$$B \int_0^{\alpha} PV[\pi(v)]j(v)\,dv. \tag{3}$$

Thus the participation decision depends on the sign of

$$\begin{aligned} H = (0.85 - \alpha)\, B y_0\, e^{-r(T-t)}[P_G N(-d_2) - F_t N(-d_1)] \\ - B \int_0^{\alpha} PV[\pi(v)]j(v)\,dv. \end{aligned} \tag{4}$$

Note that

$$\partial H/\partial \alpha < 0,\ \partial H/\partial P_G > 0.$$

We also note that

$$\partial H/\partial F_t = -(0.85 - \alpha) B y_0 e^{-r(T-t)} N(-d_1) - B \int_0^{\alpha} \{\partial PV[\pi(v)]/\partial F_t\} j(v)\, dv < 0. \quad (5)$$

As futures price rises, participation falls for two reasons: the value of the guarantee falls and the cost of set-aside rises.

4. THE PARTICIPATION DECISION

In this section, we will consider the participation decision of a representative Midwestern corn producer in 1993. There are seven principal variables, the value of which may influence the decision. These are: the set-aside rate, the target price, the sowing date futures price, a measure of land quality spread, a measure of mean land quality, implied volatility, and interest rates. Implied volatility is measured as 0.224 from the April 16, 1993 price of the September 1993 call with strike price $2.40. The interest rate is assumed to be the April 1993 prime rate of 6 percent. Three set-aside rates are considered: α = 0.05, 0.1, 0.15. Three target prices are considered: P_G = $2.60/bu, $2.75/bu, $2.90/bu. The price $2.75/bu is the target price set in law by the 1990 Farm Bill for the period 1991 through 1995. The futures price, F_t, for settlement in December 1993 is assumed to be $2.415/bu, the 4/16/'93 price.

In modeling land quality, we assume that per-acre rental rate fairly represents quality. We consider a farm with average quality land of $100/acre and another with average quality land of $120/acre. We assume profit is uniformly distributed and consider land-quality distributions on each farm type. The first quality distribution assumes all land is of the same quality; the second assumes the rental value is uniformly distributed from plus $30 to minus $30 of the mean quality. Thus the four distributions are:

1) U[$100, $100] 2) U[$70, $130]

3) U[$120, $120] 4) U[$90, $150].

All distributions are considered in Table 1. We can see that for the program parameters used in Table 1, the per-acre value of the program is almost always positive.[1]

The values in Table 1 rise with P_G, and with the futures price, and fall with the set-aside rate. They rise, but only marginally, with a rise in the spread of land quality. If one assumes that base yield rises with average land quality, then the program would appear to be approximately land quality neutral. For the conditions existing in the spring of 1993 ($2.75 target price, and 10 percent set aside), the program is worth approximately $20/acre to producers.

The approach used to derive the values in Table 1 is of direct use to producers and to extension agents. In years when the government program is less generous, producers could use this approach when deciding on program participation. Also, a slight modification of the procedure would allow the producer to solve for a critical land rental value. If participation requirements required producers to set aside land with a rental value greater than this critical value, then program participation would not be optimal. A final use for the values presented in Table 1 would be for calculating the expected returns on corn production. This would equal the futures price times the expected yield plus the program yield times the values presented in Table 1.

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[1]Possible explanations for the real world nonparticipation are cross-compliance constraints, environmental restrictions, farm payment limitations, and philosophical dislike of government programs.

Table 1: The Participation Decision: Per acre Value of the Target Price Program.

Program Parameters					
α	P_G	a) Average land quality \$100 and base yield 115 bu/acre[1]		b) Average land quality \$120 and base yield 135 bu/acre	
		Distribution			
		\$100-\$100	\$70-\$120	\$120-\$120	\$90-\$150
0.05	\$2.60	\$18.2	\$19.6	\$21.2	\$22.6
	\$2.75	\$28.7	\$30.1	\$33.5	\$34.9
	\$2.90	\$40.4	\$41.8	\$47.3	\$48.7
0.10	\$2.60	\$11.9	\$14.5	\$13.7	\$16.3
	\$2.75	\$21.7	\$24.3	\$25.2	\$27.8
	\$2.90	\$32.7	\$35.3	\$38.1	\$40.7
0.15	\$2.60	\$5.6	\$9.3	\$6.2	\$9.9
	\$2.75	\$14.7	\$18.5	\$16.9	\$20.6
	\$2.90	\$25.0	\$28.7	\$28.9	\$32.7
0.20	\$2.60	\$-0.7	\$3.9	\$-1.4	\$3.3
	\$2.75	\$7.8	\$12.4	\$8.6	\$13.3
	\$2.90	\$17.3	\$22.0	\$19.8	\$24.5

[1]To calculate the per bushel value, divide the per acre values by 115 and 135. To incorporate basis, simply adjust the futures price by the expected difference between local cash prices and the maturity futures price.

5. THE EFFECT OF MARKET COMPLETION

To evaluate the effects of market completion on welfare and output, we consider a typical Iowa crop producer. The farm is 400 acres all under crops. Corn is grown on 60 percent of the land and soybeans on the remaining 40 percent. Budget data is extracted from "Estimated Costs of Crop Production in Iowa, 1993" (Iowa State University 1992). Expected corn yield is 135 bu/acre while expected soybean yield is 45 bu/acre. Per-acre total costs for corn is $297.84, while for soybeans it is $233.76. Spring futures prices for December corn and November soybeans are $2.415 and $5.975, respectively. It is assumed that program yield per acre is the expected 1993 yield per acre. The set-aside rate and target price are set equal to the actual 1993 values of 10 percent and $2.75/bu, respectively. Annual implied volatility is assumed to be 0.2240 for corn and 0.1575 for soybeans as imputed from Black's formula and April 1993 option prices for September at the money contracts. We have no way of imputing the 1993 log correlation between corn and soybean prices, but we assume that it is positive, and use three values of 0.1, 0.5, and 0.9. We choose a CARA utility function and consider two reasonable risk aversion levels representing low, and high-risk aversion (see Babcock et al. for a description on how to choose risk aversion coefficients). Finally, we consider four levels of market completion: no futures or options markets (column 3), futures markets in both corn and soybeans (column 4), one put market with strike price $2.75 (column 6), and futures together with a put market with strike price $2.75 (column 5). We also consider

a nonrandom subsidy which generates the same expected cost for the government. We consider the impact of this nonrandom subsidy when these are no markets (column 7) and when there are both futures and options markets (column 8). This nonrandom subsidy increases producer returns by the same amount, but does not have the risk-reducing effects of the put option.

Table 2 presents the certainty equivalent returns for each situation.[2] These figures are slightly exaggerated because risk-averse producers will control risk by reducing output. It can be seen that market completion is worth about $2,700 to producers with high-risk aversion. As the correlation between corn and soybean prices rise, the markets become more useful (increased certainty equivalent returns) because the producer's revenue becomes more variable. The availability of the options market at the target price when no futures markets exist (column 6) does not increase the producer's welfare as much as the availability of futures markets only (column 4). This is because the option position the government donates to the producer is approximately correct, i.e., it is the one he or she would have purchased on the commodity options market.

Comparing the nonrandom subsidy with the target price program when no financial markets exist (columns 7 and 3), we see that while low risk averters are not substantially affected, high-risk averters would prefer by far the target price program. When futures and options markets exist, producers are indifferent

[2]The procedure we use to calculate the CER values is presented in the Appendix.

Table 2. Effect of market completion on certainty equivalent return to a typical Midwestern grain producer enrolled in the 1993 farm program

ρ (1)	Risk Aversion Level λ (2)	No Futures or Options (3)	Futures Only (4)	Futures and Options (5)	Options Only (6)	Non-random Subsidy (same ex-ante cost to government)	
						No Futures or Options (7)	Futures and Options (8)
0.1	0.00001	$64,053	$64,188	$64,227	$64,071	$63,418	$64,227
	0.0002	$61,727	$63,739	$64,227	$61,854	$51,830	$64,227
0.5	0.00001	$64,014	$64,188	$64,227	$64,078	$63,225	$64,227
	0.0002	$61,502	$63,729	$64,227	$62,138	$49,198	$64,227
0.9	0.00001	$63,974	$64,188	$64,227	$64,111	$63,031	$64,227
	0.0002	$61,368	$63,729	$64,227	$62,975	$46,791	$64,227

between the policies (columns 5 and 8).

These results indicate that in the EC, where options markets by and large do not exist, a target price program would be preferred to nonrandom direct payments. This result is consistent with Innes (1990 a, 1990 b). Curiously, in changing their crop support programs, the EC appears to have opted for a nonrandom subsidy (see any description of the CAP reform proposals). This may be because of a belief that farm prices will not vary much in the future, or it may be to make farm support more open and so easier to control.

6. THE PROGRAM AND MARKET COMPLETION

In this section, we shall present two theorems that formalize the statements made earlier about market completeness. These theorems are relevant because EC policy makers have argued that U.S. deficiency payments schemes are not production neutral and also because the ongoing reforms of the CAP are moving the EC toward U.S.-type deficiency payments schemes (FAPRI 1992).

A Separation Theorem:[3] *Consider a risk averse expected utility maximizing firm with CARA preferences facing a linear profit function and an uncertain price. Let there be no contingent markets except a put market with strike price P_G. Then output is invariant to the farm's endowment of program output, while endowments are substituted one for one with put purchases in the options market.*

Proof: Express the expected utility function as

$$EU[Py - C(y) + (P_G - P)LX_0 + ((P_G - P)L - P_Z)Z]$$
$$\text{where } L = 0 \text{ if } P_G < P \qquad (6)$$
$$L = 1 \text{ otherwise.}$$

Here y denotes output, C(y)is the cost function, X_0 is the farm's endowment of program output, Z is farm purchases of put options, and P_Z is the price of a put option. The producer chooses y and Z. The first order conditions are

[3]We call this a separation theorem because the farm program is completely decoupled under CARA.

$$E\{U'[A]\ [P - C'(y)]\} = 0, \tag{7}$$

and

$$E\{U'[A][(P_G - P)L - P_Z]\} = 0, \tag{8}$$

where $A = Py - C(y) + (P_G - P)\,L\,X_0 + ((P_G - P)\,L - P_Z)\,Z$. Completely differentiating the first order conditions with respect to y, Z, and X_0 we find

$$\frac{\partial y}{\partial X_0} = \frac{E_{Z,Z}E_{y,X_0} - E_{y,Z}E_{Z,X_0}}{E_{y,y}E_{Z,Z} - E_{y,Z}^2}, \tag{9}$$

$$\frac{\partial Z}{\partial X_0} = \frac{E_{y,y}E_{Z,X_0} - E_{y,X_0}E_{y,Z}}{E_{y,y}E_{Z,Z} - E_{y,Z}^2}, \tag{10}$$

where $E_{a,b}$ is the cross partial derivative with respect to a and b. In each case by substituting in the CARA condition and using the first order conditions we find that $E_{Z,Z} = E_{Z,X_0}$ and $E_{y,Z} = E_{y,X_0}$, and so equation (9) reduces to 0 and equation (10) reduces to - 1■

This theorem shows that the current U.S. deficiency payments scheme is decoupled in the absence of nearly all contingent markets so long as producers exhibit constant absolute risk aversion.

A Nonseparation Theorem: *Consider a risk averse expected utility maximizing firm facing an uncertain output price. Let there be no contingent markets in price, let the endowment in price guarantees be less than or equal to output, let marginal cost increase in output, and let target price be no greater than marginal cost. Then production rises with the level of endowment and with the target price.*

Proof: Denote the cumulative distribution function of price by J(P). The firm seeks to

$$Max_y \int_0^{P_G} U[Py - C(y) + (P_G - P)X_0]\, dJ(P) + \int_{P_G}^{\infty} U[Py - C(y)]\, dJ(P) \qquad (11)$$

The first order condition is

$$\int_0^{P_G} U'[M][P - C'(y)]\, dJ(P) + \int_{P_G}^{\infty} U'[N][P - C'(y)]\, dJ(P) = 0 \qquad (12)$$

where

$$M = Py - C(y) + (P_G - P)X_0$$
$$N = Py - C(y)$$

Totally differentiating (12) with respect to X_0 and y we find

$$dX_0 \int_0^{P_G} U''[M][P - C'(y)][P_G - P]\, dJ(P) +$$
$$dy \int_0^{P_G} \{U''[M][P - C'(y)]^2 - U'[M]\, C''(y)\}\, dJ(P) \qquad (13)$$
$$+ dy \int_{P_G}^{\infty} \{U''[N][P - C'(y)]^2 - U'[N]\, C''(y)\}\, dJ(P) = 0$$

Denote the second integral in (13) by A_M, and the third integral in (13) as A_N. Solving (13) for dy/dX_0, we get

$$dy/dX_0 = - \int_0^{P_G} U''[M][P - C'(y)][P_G - P]\, dJ(P)/[A_M + A_N] \qquad (14)$$

Because $C''(y) > 0$, we see that the denominator is negative. Because the integration is over the set $[0, P_G]$, we see that $P_G - P > 0$ in the numerator. Because $P_G < C'(y)$, the numerator must be negative, and so output rises with the endowment. Now totally differentiate (12) with respect to P_G and y. The two terms found using the Leibnitz rule on the bounds of integration cancel, to leave

$$dy\, A_N + dy\, A_M + dP_G\, X_0 \int_0^{P_G} U''[M][P - C'(y)]\, dJ(P) = 0$$

Tidying up we find

$$dy/dP_G = - \int_0^{P_G} U''[M][P - C'(y)]\, dJ(P)/[A_M + A_N] > 0 \blacksquare$$

This theorem tells us that under incomplete markets, output will rise with increased price support if the target price is lower than marginal cost.

7. CONCLUSIONS

By modeling the program participation decision as that of determining whether an option is fairly priced, we have developed a way of measuring the per bushel or per acre benefit to producers of any given set of program parameters. It can be shown that, because options markets exist, the U.S. deficiency payments scheme does not substantially alter planned production. It can also be shown that U.S.-type schemes encourage producers to participate in commodity options markets. In economies where no commodity options markets exist, U.S.-style deficiency payments schemes would have output expansionary effects because they reduce risk. This is a paradoxical result. U.S. producers have access to government-provided options *and* market provided options; producers in Europe do not have access to any type of options market. If, as now seems likely, the EC moves toward a U.S.-style deficiency payments scheme, the per-unit benefit to producers and the output expansionary effects will be higher than an equivalent scheme in the United States.

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APPENDIX I

Consider the CARA utility function U(Y(P)] where Y, income, depends on price, P.

$$U[Y(P)] = 1 - e^{-\lambda Y(P)}$$

where λ is the risk aversion coefficient. Income will be the sum of business and financial profits. Using GAUSS software, we numerically integrate utility with respect to the price density function

$$\int_{o}^{\infty} (1-e^{-\lambda Y(P)})\, dP = EU.$$

Denote the certainty equivalent return by c. It is the certain income that generates the same utility as a risky income distribution.

$$1 - e^{-\lambda c} = EU$$

$$\therefore c = \frac{Ln[1 - EU]}{(-\lambda)}$$

To get an intuitive understanding of this procedure, draw a risk-return indifference curve. The CER is the point at which this curve intersects the vertical axis.

PAPER II

GOVERNMENT COSTS OF TARGET PRICE SUPPORTS

GOVERNMENT COSTS OF TARGET PRICE SUPPORTS

by

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ABSTRACT

The U.S commodity target price program can be considered as the issuance by the U.S. government of contingent claims to participating farmers. This paper models the acreage set-aside requirement as the premium paid for these contingent claims. A computable, equilibrium model of the interrelationships between program parameters, production, futures price, program cost, and producer benefit is developed. The model can accommodate stochasticity on either supply and\or demand. We use the model to estimate the expected government cost of the corn target price program in 1993.

1. INTRODUCTION

The U.S. government uses a target-price, deficiency-payments scheme to support the income of most corn and wheat producers. The government guarantees to offset the revenue implications of a price outcome below a politically determined price floor. This price floor is called the target price. Since 1986, the guarantee relates only to a prespecified volume of output. It applies only if the producer agrees to leave idle a prespecified area of land. Thus, the benefit of program participation is a guaranteed price floor, while the cost of participation is the profit foregone due to the set-aside requirement.

Gardner (1977) observed the similarity between the program and yet to be developed agricultural options contracts. Kahl (1986) suggested that it would be more efficient for the government to use options markets to stabilize producer income than to continue writing its own de facto put contracts. The 1990 Food, Agriculture, Conservation, and Trade Act legislated for the direct use of traded options markets as Kahl had suggested. This Options Pilot Program is available to producers as an alternative to existing target price and loan programs for some commodities in nine midwestern counties for the 1993 crop year.

In this paper, we first show that the program participation decision is a straightforward cost benefit comparison that does not involve the individual's attitude toward risk. Because the participation decision is deterministic, we can impute, for any choice of target price and set-aside requirements, whether an individual will participate. Some additional knowledge of how producers will

respond allows us to predict nationally how much area will be set aside for any choice of program parameters. This in turn allows us to predict actual production, and the futures market rational expectations equilibrium response to this expected production. The model is closed by the direct link between the futures price and the option value. Model closure allows us to predict government costs, futures prices, and production for any given choice of program parameters. The last section of the paper implements the procedure using actual 1993 data.

Numerous simplifying assumptions are required to close the model and implement the procedure. These assumptions are used to overcome a lack of farm-level data. This data would be expensive to construct, but should be worthwhile given the ex ante farm program costs that can be measured. In the absence of this farm-level data, the procedures reported below will give a less accurate but, hopefully, unbiased measure of these costs.

2. PREVIOUS WORK

There has been considerable analysis on the impact of government program parameters on crop supply. These include Floyd (1965), Lidman and Bawden (1974), Evans (1980), Lee and Helmberger (1985), Miranda and Helmberger (1988), and Perry et al. (1989). Of these studies, only one, Lee and Helmberger, attempted to model how supply would respond to a free market regime. This is because supply control of one form or another has been in effect for most major crops in the United States for all but a handful of years since the second world war. Gardner (1987) considered some of the political determinants of the nature and magnitude of agricultural support. Chavas et al. (1983) examined the acreage response to program parameters and futures prices. Marcus and Modest (1984) studied the input decision in an uncertain output price and quantity environment under a target price regime. In a subsequent paper, Marcus and Modest (1986) investigated the cost to the government of the price floor. None of these papers attempt to model the simultaneity of the interaction between the planting decision and the futures price at planting.

3. THE CONTINGENT CLAIMS MODEL

Under the provisions of the 1990 Farm Act, a participant contracts to idle a certain fraction, α, of base acres (eligible acres), B, in return for a price floor guarantee, P_G, on a prespecified eligible per acre output, y_0, for each of (1 - 0.15 - α)B acres. The 15 percent of base acres on which no deficiency payment is paid may be planted under any crop except fruits and vegetables. Assume that there is easy access to put markets, and denote the present value of a put option with strike price P_G as $W(P_G,F_{c,t})$ where $F_{c,t}$ is the price at time t of a harvest date corn futures contract. Because the producer can sell puts with strike price P_G to restore the position that would have pertained were there no program, the dollar value of the program benefits is

$$(0.85 - \alpha)\, B\, y_0\, W(P_G,F_{c,t}). \tag{1}$$

We make the Black and Scholes assumptions (1973) concerning the economic and trading environment. The benefit of the program is identical to the cost of (0.85 - α) B y_0 puts expiring at the harvest date. Thus, using the standard value of a put option (Rubinstein, 1976), the program benefits are

$$(0.85 - \alpha)\, B\, y_0\, e^{-r(T-t)}\, [P_G\, N(-d_2) - F_{c,t}\, N(-d_1)] \tag{2}$$

where t is the time index;

t = T is time index value at harvest;

$t = 0$ is time index value at sign up for the program;

$N(.)$ is the cumulative density function of a standard normal distribution;

$d_1 = [\ln(F_t/P_G) + 0.5\ \sigma^2\ (T - t)]/\sigma\ (T - t)$;

$d_2 = [\ln(F_t/P_G) - 0.5\ \sigma^2\ (T - t)]/\sigma\ (T - t)$;

r is the prime interest rate;

σ is the implied volatility or the annualized standard deviation of the log of futures price.

The cost is the present value of foregone profit from set-aside land. The idled land will be the 100 α percent of lowest economic quality land[4]. Let PV(.) denote the present value operator, let $\pi(v)$ denote per acre profit from land of quality v, let v be indexed such that $0 \leq v \leq 1$, and let j(v) be mass density function of v. Then the cost is

$$B\int_0^{\alpha} PV[\pi(v)]\ j(v)\ dv. \qquad (3)$$

If contingent markets are unbiased and complete, then on any part of land $PV[\pi(v)]$ profit can be assured. Thus, the participation decision depends on the sign of

[4] The land set aside might not be of the lowest quality because factors such as accessibility and field size may enter the set-aside decision (Hoag et al. 1993).

$$H = (0.85 - \alpha)\, B\, y_0\, e^{-r(T-t)}[P_G\, N(-d_2) - F_t\, N(-d_1)] \\ -B \int_0^{\alpha} PV[\pi(v)]\, j(v)\, dv. \tag{4}$$

Note that

$$\partial H/\partial \alpha < 0,\ \partial H/\partial P_G > 0.$$

Having considered the participation decision of an individual producer we will now aggregate up to the market level.

4. THE MARKET MODEL

In this section, we develop a closed general equilibrium model of corn and soybean production, and of participation in the corn target price program. Inevitably, several simplifying assumptions are necessary. We assume that there is a fixed stock of land, A_0. We set this equal to 143 million acres, the average amount of land either under corn or soybeans, or set aside to corn in the crop years 1989/90, 1990/91, and 1991/92. This land is assumed to have only two uses, corn and soybean production. While not strictly correct, this assumption is not far from reality[5]. Let A_{sb} and A_p denote acres in soybeans, and total acres in the program respectively. Let A_c denote acres in corn, but not in the program, surplus to the 0.15 A_p in the program on which no deficiency payment is made. Thus, A_c could be negative, but could never have value less than - 0.15 A_p. Then

$$A_p = A_0 - A_{sb} - A_c. \tag{5}$$

The expression for total acres in corn is

$$A_{tc} = A_c + (1 - \alpha)A_p. \tag{6}$$

We propose a technology

$$Q_i = k_i A_i^{d_i+1}, \tag{7}$$

where Q_i is total output of crop i; i = tc, sb; and $k_i A_i^{d_i}$ is yield per acre. Note

[5] To accurately implement the procedure we propose here, data would be needed to partition farms by size, base acreage allotment, region, land quality, and cost of production.

that 100 x d_i is the percent change in yield per acre associated with a percent change in acres sown. In the agricultural economics literature, d_i is commonly called the slippage coefficient. Henceforth we make the assumption that the coefficient is the same for both crops.

Our ultimate intention is to find the rational expectations equilibrium futures prices and planted acres when output is stochastic. However, in order to develop intuition, we will first find the rational expectations equilibrium when output is deterministic and uncertainty enters through demand. We assume that per bushel variable costs, b_c and b_{sb}, are constant.

Because there is a limit to the amount of land that can be placed in the program, the optimization problem involves two steps. First, we maximize the national profit function which is

$$\pi = (0.85 - \alpha) A_p y_0 W(P_G, F_{c,t}) + \{F_{c,t} e^{-r(T-t)} - b_c\} k_c [A_c + (1 - \alpha) A_p]^{1+d} + \{F_{sb,t} e^{-r(T-t)} - b_{sb}\} k_{sb} [A_0 - A_c - A_p]^{1+d}. \tag{8}$$

Maximizing (8) with respect to A_c and A_p, we get

$$\frac{\partial \pi}{\partial A_c} = \{F_{c,t} e^{-r(T-t)} - b_c\} k_c [A_c + (1 - \alpha) A_p]^d (1 + d) - \{F_{sb,t} e^{-r(T-t)} - b_{sb}\} k_{sb} [A_0 - A_c - A_p]^d (1 + d) = 0, \tag{9}$$

$$\frac{\partial \pi}{\partial A_p} = \{F_{c,t}\, e^{-r(T-t)}] - b_c\}k_c\, [A_c + (1 - \alpha)A_p]^d\, (1 - \alpha)(1 + d)$$
$$- \{F_{sb,t}\, e^{-r(T-t)} - b_{sb}\}k_{sb}[A_0 - A_c - A_p]^d(1 + d) \qquad (10)$$
$$+ (0.85 - \alpha)y_0\, W(P_G, F_{c,t}) = 0.$$

Solving, we find

$$A_{tc} = A_c + (1 - \alpha)A_p = \left(\frac{G_2}{G_3}\right)^{1/d}, \qquad (11)$$

and

$$A_{sb} = \left(\frac{G_1 G_2}{G_3}\right)^{1/d} \qquad (12)$$

where

$$G_1 = \frac{\{F_{c,t}\, e^{-r(T-t)} - b_c\}k_c}{\{F_{sb,t}\, e^{-r(T-t)} - b_{sb}\}k_{sb}}, \qquad (13)$$

$$G_2 = (0.85 - \alpha)y_0\, W(P_G, F_{c,t}), \qquad (14)$$

and

$$G_3 = \{F_{c,t}\, e^{-r(T-t)} - b_c\}\, k_c\, \alpha\, (1 + d). \qquad (15)$$

Therefore,

$$Q_c = k_c \left(\frac{G_2}{G_3} \right)^{(1+d)/d}, \tag{16}$$

and

$$Q_{sb} = k_{sb} \left(\frac{G_1 G_2}{G_3} \right)^{(1+d)/d}. \tag{17}$$

Now we have the indirect supply functions. The fraction G_2/G_3 may be interpreted as the ratio of program value to the producer to program cost to the producer. If, as we expect, $-1 < d < 0$, then both outputs will fall with a rise in this ratio. To close the model, we must establish demand relationships.

5. THE MARKET MODEL WHEN DEMAND IS UNCERTAIN

If we assume a lognormal distribution with a known implied volatility, then all variables in the expressions G_1, G_2, and G_3 are known at planting. However, $F_{c,t}$ and $F_{sb,t}$ are endogenous in that they in turn depend on the planting decision. Because $W(P_G,F_{c,t})$ depends on $F_{c,t}$, it also is endogenous. We must close the model by considering the demand side.

In their innovative paper, Marcus and Modest (1986) develop a partial equilibrium model to value the ex ante cost of the target price program. The model is partial in that it does not take into account either the aggregate effect of output decisions on the futures price, or the effect of the set-aside rate on the futures price. They assume that demand has a logarithmic relationship with both harvest date price and the Standard and Poor's 500 index. For realism here, we replace the S&P index with an index of livestock numbers, but otherwise adopt the Marcus and Modest assumptions. The resulting demand function is where QD_i is the demand for good i; $F_{i,T}$ is the price of good i at time T; S_T is the livestock index at time T; c_i is a constant; and γ_i, ξ_i are elasticities. The value for γ_i must be consistent with the volatility of futures prices. Given that at harvest supply is fixed, any change in the value of S_T will change $F_{i,T}$ so that (18) holds.

Taking the log of both sides of (18), and using the relationship

$$E[x^2] = \{E[x]\}^2 + Var[x]$$

we get

$$\gamma_i^2 \sigma_S^2 - \xi_i^2 \sigma_{i,F}^2 = 0, \tag{18}$$

where σ_S, σ_i are the implied volatility of livestock numbers on feed and the implied volatility of the harvest price of crop i, respectively. Rearranging and taking the square root of this expression gives

$$\gamma_i = \frac{\xi_i \ \sigma_{i,F}}{\sigma_S}. \tag{19}$$

Now, to develop a relationship from the demand side between futures price and livestock numbers, note that futures price depends only on time and livestock numbers,

$$F_t = F(S_t).$$

We assume that livestock numbers follow a geometric Brownian motion. Thus,

$$dS_t = S_t \ \sigma_s \ dz_s \tag{20}$$

and,

$$dz_s = \mu_{S,t} \ (dt)^{0.5}. \tag{21}$$

The term $\mu_{S,t}$ is a random variable with standard normal distribution and z_S is the standard Wiener process. Now we can apply Ito's lemma, and adjust for the rate of return, to find

$$\frac{\partial F}{\partial t} + 0.5\, S^2\, \sigma_s^2\, \frac{\partial^2 F}{\partial S^2} + r\, S\, \frac{\partial F}{\partial S} = 0. \tag{22}$$

For a comprehensive explanation of how this equation was arrived at, see Black (1976). Equation (22) is the law of motion describing how futures price changes as livestock numbers change, and as time changes. The boundary value conditions are

$$QD_i(F_{i,T}, S_T) = c_i\, S_T^{\gamma_i}\, F_{i,T}^{-\xi_i} \tag{23}$$

$$QD_i(F_{i,T}, 0) = 0. \tag{24}$$

The first boundary condition states that the futures price at settlement must be consistent with the demand equation. The second states that if livestock numbers are zero, then the livestock economy has collapsed and there is no demand. Solving (22) subject to (23) and (24), we find

$$QD_i = c_i\, S_t^{\gamma_i}\, F_{i,T}^{-\xi_i}\, e^{[\gamma_i r + 0.5(\frac{\gamma_i}{\xi_i})(\gamma_i - \xi_i)\sigma_S^2](T-t)}. \tag{25}$$

At any time between planting and harvesting, the futures price $F_{i,t}$ must be consistent with the index, S_t, and the quantity QD_i[6]. The expression

[6] The problem may also be viewed as a discrete process with two dates of interest. These are the date that planting and hedging occurs, and the date that harvesting and hedge lifting occurs. From this perspective, output is not a geometric Brownian process, but rather an initial condition at planting that has an associated lognormal distribution at harvest.

$$e^{[\gamma_i r + 0.5(\frac{\gamma_i}{\xi_i})(\gamma_i - \xi_i)\sigma_S^2](T-t)},$$

is an actuarially fair premium on the futures price due to price uncertainty and the nonlinearity of demand. Fischer (1975, p 513) provides a good intuitive explanation of how this phenomenon arises from Jensen's inequality.

We now have supply, (16) and (17), and demand equations, (25), for both corn and soybeans. As the supply decision is taken at planting (t = 0), the equations should be solved at t=0. Using (5), (11), and (12), we can now work back to find the equilibrium acreage allocation (A_c, A_{sb}, A_p). Finally, we have the expected cost to government

$$CG = (0.85 - \alpha) y_0 A_p W(P_G, F_{c,t}^*), \tag{26}$$

where $F_{c,t}^*$ is the rational expectations equilibrium corn futures price.

6. THE MARKET MODEL WHEN SUPPLY IS UNCERTAIN

In the previous section, we have shown how to value government cost when demand is uncertain. One of the more distinctive features of agricultural production is the uncertainty of supply. In this section, we will extend the model to incorporate output uncertainty[7].

Like the settlement price, output is observed only once. However, through the futures price, the market makes explicit its expectations of settlement price. This does not occur for output. Nonetheless, there are analysts capable of summarizing meteorologic and other data into a stochastic process describing probable harvest output. For a more empirical discussion on the relationship between the crop output distribution and the price distribution, see Thompson (1982, 1986).

The stochastic process describing harvest output is denoted by $Q_{i,t}$ where t denotes a time index initialized at 0 for planting and T for harvest date. We assume that the process has initial conditions described by the number of acres planted

$$Q_{i,0} = k_i A_i^{d_i+1}. \tag{27}$$

Here $Q_{i,0}$ is the exponential of the expected value of log of harvest output. Equation (27) may alternately be viewed as the mode of the lognormal output

[7] The model can be solved with both supply and demand uncertainty but the only new element, correlation between the shocks, does not justify the amount of math involved.

distribution. It then exhibits geometric Brownian motion with zero drift, i.e.

$$dQ_i = Q_i \, \sigma_{i,Q} \, dz_{i,Q}. \tag{28}$$

The constant $\sigma_{i,Q}$ in (28) is the annualized standard deviation of the instantaneous rate of change in expected output while

$$dz_{i,Q} = \mu_{i,t} \, (dt)^{0.5}, \tag{29}$$

where $\mu_{i,t}$ is a random variable with standard normal distribution, and $z_{i,Q}$ is the standard Wiener process. We must now distinguish between actual supply, $Q_{i,T}$, and the initial condition, $Q_{i,0}$.

From the producer's perspective, the initial condition is the only control variable. He/She can choose only A_i which, through (27), determines $Q_{i,0}$. The supply side relationship previously established between $Q_{i,0}$ and $F_{i,0}$ in equations (16) and (17) does not change. This is because at t = 0, (16) and (17) represent output predictions given the number of acres planted. At planting time, there is a perfect mapping between acres planted and expected output. At later times, because of output uncertainty, this relationship ceases to be perfect. To establish the demand side relationship, we need first to value the crop.

Suppressing S_T in (18), we find

$$QD_i(F_{i,T}) = c_i \, F_{i,T}^{-\xi_i}.$$

Thus, at harvest the value is

$$V_i(F_{i,T}) = c_i \, F_{i,T}^{1-\xi_i} \tag{30}$$

where $V_i(.)$ is the value of crop i.

One of the consequences of assuming that absolute own-price demand elasticities are constant and less than one, is that the value of output is zero when the futures price is zero. If the absolute values of demand elasticities were greater than one, this condition would not hold. Thus, we have

$$V_i(F_{i,t} | F_{i,t} = 0) = 0, \tag{31}$$

i.e. the value of the crop when $F_{i,t} = 0$. Note that because output is assumed to be lognormally distributed, and because the own-price elasticity is constant, futures price must also be lognormally distributed. Specifically, futures prices will evolve over time according to geometric Brownian motion. If the standard deviation of the log of futures price is $\sigma_{i,F}$, then

$$\sigma_{i,F} = \frac{\sigma_{i,Q}}{\xi_i}. \tag{32}$$

This can be seen from the reasoning behind (19). An Ito's lemma expansion of $V_i(F_{i,t})$ using Black and Scholes method gives

$$\frac{\partial V_i}{\partial t} + 0.5 \, F_i^2 \, \sigma_{i,F}^2 \, \frac{\partial^2 V}{\partial F_i^2} - r \, V_i = 0. \tag{33}$$

When we solve this and impose the boundary value conditions, (30) and (31), we get

$$V_i(F_{i,t}) = c_i F_{i,t}^{1-\xi_i} e^{[-0.5\xi_i(1-\xi_i)\sigma_{i,F}^2 - r](T-t)}. \quad (34)$$

To develop a relationship between futures price and output, when output is lognormally distributed, assume that individual output is strictly proportional to national output[8]. From the geometric Brownian motion of Q_i and from the functional dependence,

$$F_{i,t} = F_{i,t}[Q_{i,t}]. \quad (35)$$

We apply Ito's lemma to get

$$\frac{\partial F_i}{\partial t} + 0.5\, Q^2\, \sigma_{i,Q}^2 \frac{\partial^2 F_i}{\partial Q_i^2} = 0. \quad (36)$$

As in equation (22), this is the law of motion describing how futures price changes with the stochastic process for output, and with time. This is Black's (1976) partial differential equation except for the term $r F_i$ which is missing because, unlike price, output does not have a trend due to interest rates.

With boundary value condition

[8] We could relax this assumption to: the log of individual output is perfectly linearly correlated with the log of national output. Losq (1982) makes this assumption. Alternately, we could permit imperfect correlation by incorporating CAPM risk pricing. Both relaxations complicate the model unduly.

$$F_{i,T} = \left(\frac{Q_{i,T}}{c_i}\right)^{-1/\xi_i},$$

and using $\sigma_{i,F} = \sigma_{i,Q}/\xi_i$, we solve to get

$$Q_{i,t} = c_i F_{i,t}^{-\xi_i} e^{0.5\xi_i(1+\xi_i)\sigma_{i,F}^2(T-t)}. \tag{37}$$

Evaluate at t = 0 (planting time) to get the mode of the output distribution, $Q_{i,0}$. From (27) we know that the producer has control of the mode of output. Divide crop value (equation (34)) by $Q_{i,0}$ to get a value (price) per unit of this concept of output,

$$F_{i,0} e^{[-\xi_i\sigma_{i,F}^2 - r]T}. \tag{38}$$

This price is lognormally distributed with variance of the logs equal to $\sigma_{i,Q}^2/[\xi_i]^2$. All the uncertainty has now been transferred, symbolically, onto price.

Expression (38) can be viewed as a shadow price; the price a risk neutral or well-hedged risk averse individual responds to when allocating land to crop i. If $\sigma_{i,F}^2 > 0$, then the shadow price is less than the discounted expected futures price. This is because $\xi_i > 0$ implies a negative correlation between output and price, and so the expected revenue is less than the product of expected price and expected output. When there is no output uncertainty, then the producer responds to the price $F_{i,0} e^{-rT}$, which arises in (13), (14), and (15).

We return to equations (13), (14), and (15) in the production decision.

Evaluating at planting and replacing $F_{i,0}\, e^{-rT}$ with (38), we map

$$G_1 \to N_1 = \frac{\{F_{c,0}\, e^{[-\xi_c \sigma_{c,F}^2 - r]T} - b_c\}k_c}{\{F_{sb,0}\, e^{[-\xi_{sb} \sigma_{sb,F}^2 - r]T} - b_{sb}\}k_{sb}}, \tag{39}$$

$$G_2 \to N_2 = (0.85 - \alpha)\, y_O\, W(P_G, F_{c,0}), \tag{40}$$

and

$$G_3 \to N_3 = \{F_{c,0}\, e^{[-\xi_c \sigma_{c,F}^2 - r]T} - b_c\}k_c\ \alpha\ (1 + d). \tag{41}$$

Note that the formula for $W(P_G, F_{c,0})$ as expressed in (2) does not change, because the quantity of puts granted by the government is fixed. We complete the system as before to get expressions for $Q_{c,0}$ in terms of $F_{c,0}$ and $Q_{sb,0}$ in terms of $F_{sb,0}$. We now have four equations in four unknowns, $\{Q_{c,0}, F_{c,0}, Q_{sb,0}, F_{sb,0}\}$. We solve the system

$$Q_{c,0} = k_c \left(\frac{N_2}{N_3}\right)^{(1+d)/d}, \tag{42}$$

$$Q_{sb,0} = k_{sb} \left(\frac{N_1 N_2}{N_3}\right)^{(1+d)/d}, \tag{43}$$

$$Q_{c,0} = c_c\, F_{c,0}^{-\xi_c}\, e^{0.5\xi_c(1+\xi_c)\sigma_{c,F}^2 T}, \tag{44}$$

$$Q_{sb,0} = c_{sb} F_{sb,0}^{-\xi_{sb}} e^{0.5\xi_{sb}(1+\xi_{sb})\sigma_{sb,F}^2 T}, \tag{45}$$

by substituting out $Q_{c,0}$ and $Q_{sb,0}$, and then using numerical methods (GAUSS 1988). Then we find $\{A_c, A_s, A_p\}$ from (46), (47), and (48).

$$A_p = A_0 - A_{sb} - A_c. \tag{46}$$

$$A_{tc} = A_c + (1 - \alpha)A_p = \left(\frac{N_2}{N_3}\right)^{1/d}. \tag{47}$$

$$A_{sb} = \left(\frac{N_1 N_2}{N_3}\right)^{1/d}. \tag{48}$$

The equation for cost to government, CG, is (26) as given previously.

7. IMPLEMENTATION

We have yet to specify the model's parameters. In a survey of the literature, Hoag et al. (1993) reported that estimates of slippage coefficients vary from - 0.25 to - 0.58. We will assume that it is equal to - 0.35 for both crops. The k_i are productivity coefficients and will change over time. Their value was arrived at using a log-log restricted seemingly unrelated regression system, the details of which are outlined in Appendix 1. Adjusted to 1993, the estimated coefficients are $k_c = 4.5915 \times 10^8$ and $k_{sb} = 1.375 \times 10^8$, where output is measured in bushels and area is measured in million acres. We find c_c, and c_{sb}, the constant terms in the demand equations, using a log-log restricted seemingly unrelated regression system. The system is outlined in more detail in Appendix 1. The estimated time and inflation adjusted coefficients are $c_c = 9.7316 \times 10^9$ and $c_{sb} = 3.6858 \times 10^9$. We assume that variable costs, b_c and b_{sb} are constant and equal to those available for Iowa in 1993. These costs are $1.08 per bushel for corn and $2.11 per bushel for soybeans[9]. The per acre base yield is assumed to be the national average of 105 Bu/acre (U.S. Feed Grain Council 1993). We assume that there are 76.62 million effective corn base acres[10].

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[9] "Estimated Costs of Crop Production in Iowa, 1993." Iowa State University, University Extension, Ames, IA.

[10] The American Soil Conservation Service (ASCS) reports corn base acres for 1993 as 82.2 m acres (USDA 1993). However, over previous years the maximum participation rate was 90.5 percent in 1987/88. To allow for the participation disincentive of payment limits, we assume a maximum participation rate of 92 percent. Payment limits are $50,000 and $75,000 per farm for deficiency payments and total payments respectively.

Marcus and Modest assume a zero basis and use, as we do, absolute own-price demand elasticities of 0.3 and 0.4, respectively, for corn and soybeans. These figures are from George and Kirby (1971) and are consistent with more recent estimates (Rojko et al. 1978; Collins 1985). The implied volatilities, $\sigma_{c,F}$ and $\sigma_{sb,F}$, are assumed to be 0.2241 for corn and 0.1575 for soybean as imputed from the Black formula and April 16, 1993 option prices for September at the money contracts. The interest rate is assumed to be 0.06 (6 percent), the prime rate in April 1993.

We solved the model for CG as the government parameters (α, P_G) change. The results are presented in Figure 1. There are three regions: for Region 1, P_G for any α is so low that no one participates in the program; Region 2 is where some, but not the maximum, allowable acres are planted; and Region 3 where P_G for any α is so high that all base acres are signed into the program. Notice how, at a given set-aside rate, cost rises at an increasing rate with P_G. This is because when everyone has entered the program, an increase in P_G no longer reduces plantings. Reduced plantings increase the futures price which, in turn, reduces the government cost.

Using the actual spring 1993 target price of $2.75 and set-aside rate of 10 percent, we estimate the spring 1993 expected cost of the corn program to the government to be $3.606 billion. This value is indicated on Figure 1. To the

extent that our assumptions are valid, the government could use Figure 1 to consider the costs of alternative program parameters. Table 1 contains some of the model output.

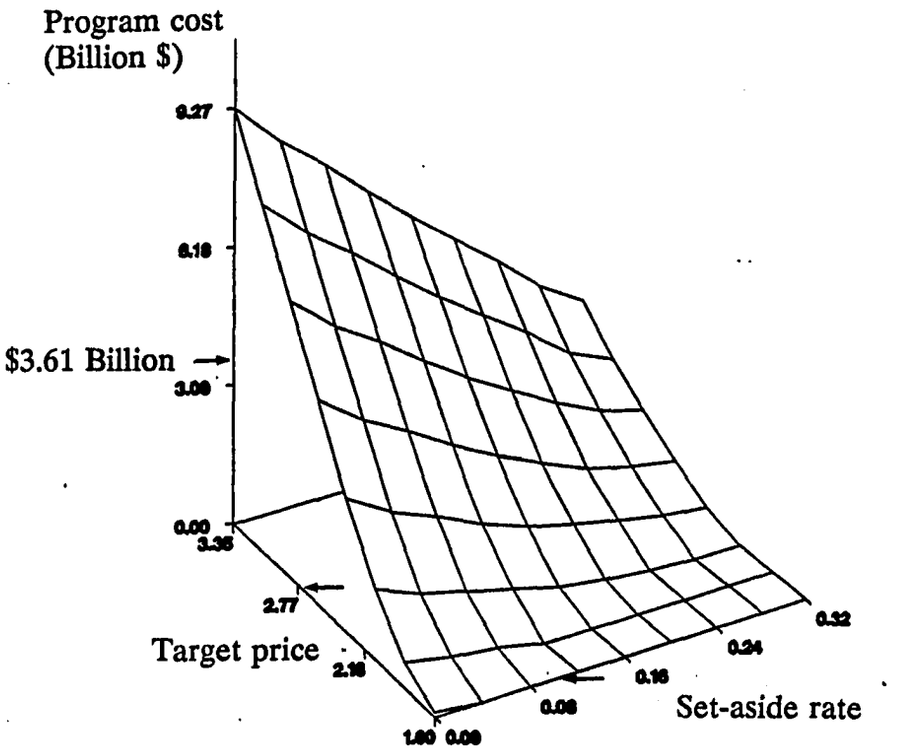

Figure 1. Expected government cost of corn target price program.
NOTE: Arrows show 1993 program parameters and expected costs.

Table 1. Summarized model output

α	P_G	CG	F_c	Q_c	F_s	Q_s	A_p	A_c	A_s
	$	Bill'n $	$	Bill'n bushels	$	Bill'n bushels	Mill'n acres	Mill'n acres	Mill'n acres
0.00	1.60	0.102	1.94	8.01	4.63	2.00	75.62	5.67	61.71
	2.10	1.483	1.94	8.01	4.63	2.00	75.62	5.67	61.71
	2.60	4.384	1.94	8.01	4.63	2.00	75.62	5.67	61.71
	3.10	7.631	1.94	8.01	4.63	2.00	75.62	5.67	61.71
0.08	1.60	0.00	1.94	8.01	4.63	2.00	0.00	81.29	61.71
	2.10	0.829	2.08	7.84	5.05	1.93	75.62	9.20	58.18
	2.60	3.137	2.08	7.84	5.05	1.93	75.62	9.20	58.18
	3.10	6.037	2.08	7.84	5.05	1.93	75.62	9.20	58.18
0.16	1.60	0.00	1.94	8.01	4.63	2.00	0.00	81.29	61.71
	2.10	0.314	2.09	7.86	5.08	1.93	33.65	51.15	58.19
	2.60	2.018	2.26	7.66	5.55	1.86	75.62	12.39	54.99
	3.10	4.485	2.26	7.66	5.55	1.86	75.62	12.39	54.99
0.24	1.60	0.00	1.94	8.01	4.63	2.00	0.00	81.29	61.71
	2.10	0.056	1.97	8.00	4.74	1.98	4.54	77.92	60.54
	2.60	1.157	2.36	7.56	5.86	1.82	62.58	26.79	53.43
	3.10	3.128	2.45	7.47	5.86	1.79	76.62	15.63	51.75
0.32	1.60	0.00	1.94	8.01	4.63	2.00	0.00	81.29	61.71
	2.10	0.00	1.94	8.01	4.63	2.00	0.00	81.29	61.71
	2.60	0.741	2.22	7.70	5.46	1.87	33.84	53.60	55.56
	3.10	1.964	2.57	7.37	6.45	1.75	66.44	26.43	50.13

8. CONCLUSIONS

U.S. grain policies have recently changed so that deficiency payments are paid on historical rather than actual production. This virtual decoupling of production and policy allows us to place an exact value on the benefits to producers. This value depends on the current price of a put option with strike price equal to the announced target price. The cost of the program to producers is the income foregone on the acreage which must be set aside to meet program requirements. This cost will vary from producer to producer. Quantifying the cost requires information on the individual's land quality distribution. If this information were available, and if one assumed that producers make rational participation decisions, then one can predict the sign-up rate for any given set of program parameters. When producers join the program, they agree to take land out of production, which in turn reduces national production. Expected changes in national production influence the futures price, and in turn the benefits of program participation (through the option price). We have shown that the system described above is closed and, given the technologic and distribution assumptions made, that there is a unique set of outputs, prices, and participation rates for each set of program parameters. Closure allows us to calculate the ex ante cost to the government of any given set of program parameters. We demonstrate the procedure using 1993 price data and we show how expected government cost increases with an increase in the target price, and decreases with an increase in the set-aside rate.

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APPENDIX I

Using USDA annual data (U.S. Feed Grains Council, 1993) for the twelve crop years 1980/81 to 1991/92, we estimate the log-log production system arising from (7) to get

$$\text{Ln}(Q_c) = 19.805 + 0.01076\ t + 0.65\ \text{Ln}(A_c) + e_c \qquad (49)$$

$$(194.8) \quad (0.78)$$

$$\text{Ln}(Q_{sb}) = 18.6 + 0.010701\ t + 0.65\ \text{Ln}(A_{sb}) + e_{sb} \qquad (50)$$

$$(351.9) \quad (1.49)$$

where

Q_i = national output for crop i (bushels); A_i = land sown to crop i (m acres);

e_i = error term for crop i; and t = year - 1980.

Adjusted for time (adjusting to 1993), we arrive at the coefficients k_c, and k_{sb} in the text. For demand, we again use USDA annual data (U.S. Feed Grains Council, 1993) for the twelve crop years 1980/81 to 1991/92 to estimate the log-log system arising from (18) with S_T suppressed

$$\text{Ln}(QD_c) = 23.141 + 0.010201\ t - 0.3\ \text{Ln}[F_{c,T}/CPI] + u_c \qquad (51)$$

$$(192.8) \quad (0.72302)$$

$$\text{Ln}(QD_{sb}) = 22.61 + 0.0097907\ t - 0.4\ \text{Ln}[F_{sb,T}/CPI] + u_{sb} \qquad (52)$$

$$(354.5) \quad (1.3578)$$

where

QD_i = total use of crop i (bushels); $F_{i,T}$ = crop i farm price at harvest

($/bu);

CPI = consumer price index (1985 = 1.00); u_i = error term for product i; and

t = year - 1980.

Adjusting for time, we arrive at the coefficients c_c, and c_{sb} in the text.

PAPER III

POLYNOMIAL PRICE CONTRACTS

POLYNOMIAL PRICE CONTRACTS

by

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1993

ABSTRACT

Evidence is presented that suggests options markets are not optimal. An alternative market structure is proposed that would increase hedging effectiveness, and the risk return tradeoff for hedgers and speculators, respectively. Fewer derivative markets would be required per underlying asset than with options markets. The settlement price of these alternative markets would be some power of the closing futures price.

1. INTRODUCTION

To accommodate movements in the price of the underlying asset, options markets, as they are currently structured, require trading for both puts and calls at several strike prices. The need for multiple interdependent options markets per asset reduces liquidity, and by extension the number of markets that can be covered by options. Were it possible to replace these multiple options markets with a single independent derivative, then liquidity and market coverage would expand. For example, at any one time more than 20 crude oil futures trade, with expiration months stretching out three years. Active options markets typically operate only on the three closest to maturity futures, with, at most, six active strike prices per contract. Therefore, although 36 separate options prices are quoted on crude oil, options do not trade on most crude oil contracts.

Among those options that do trade, measured implied volatilities have been found to relate in a systematic way to the time to maturity and degree of moneyness (Whaley 1982; Day and Lewis 1988; Stephen and Whaley 1990; Choi and Shastri 1989; MacBeth and Merville 1979). One possible explanation for the mispricing of out of the money, and far from maturity options, may be the relatively thin markets that exist for these contracts.

The markets response to the problem outlined above is the development of customized options. These options are written by brokerage houses to satisfy the individual needs of large investors, and have been so successful in drawing

liquidity from standardized options markets, that the Chicago Board Options Exchange (CBOE) has initiated a program to customize contracts for other investors (Wall Street Journal 1993a). Over the first three months of trading, they have proven to be very successful (Wall Street Journal 1993b).

One alternative to customization of options contracts would be a contract whose settlement price is some (power) function of the underlying asset price, rather than the difference between the asset price, and some arbitrary strike price as is currently used. For reasons that will later become obvious, we term these alternatives *polynomial contracts*. The purpose of this paper is to explore polynomial contracts as an alternative (or supplement) to existing options. One benefit of polynomials is the one-to-one linkage between the polynomial price and the asset price, thereby eliminating the need for multiple strike prices, and separate put and call markets. A second benefit is the replacement of a market structure that forces a discrete choice among strike prices, with a structure that provides a more flexible approximation to the optimal pay-off function.

We begin with a description of polynomial contracts. Then we show that polynomial contracts evolve as a theoretically acceptable solution to a maximization problem where hedgers and speculators maximize utility by choosing among market structures, rather than the more traditional approach where agents choose optimal positions given a fixed market structure. Then, by means of Monte Carlo experimentation, we show that markets in $P^{0.5}$ and $P^{1.5}$ of maturity prices come very close to spanning the state space. We also show the

improvement in welfare (by means of certainty equivalent returns) that results from the move from options to polynomial functions. Finally, we provide a fair pricing formula for a class of polynomial contracts.

2. PREVIOUS WORK

There exists an enormous body of literature concerning hedging with futures and options. Some of the work used mean-variance analysis (Wolf 1987). Hedging under the less structured expected utility framework has also been considered (Lapan et al. 1991; Benninga et al. 1984). Benninga et al. derived the optimum hedge ratio when today's futures price is an unbiased predictor of the futures price in the next period, and when the basis is independent of the spot price. Lapan et al. showed that under the Benninga et al. conditions, there was no hedging demand for options. However, if either futures or options are biased, then optimal hedging may involve options. Moschini and Lapan (1993) showed that options may be used when there are quasi-fixed inputs.

There is another literature that deals with the adequacy of existing hedging instruments. Arrow (1964), in his seminal paper on risk bearing, established the concept of a complete set of contingent claims markets. Such completeness exists when a payout in any state, and only in that state, is possible. In the absence of transaction costs, this completeness is a necessary pre-condition for efficient equilibrium to exist under uncertainty. A set of options markets with a continuum of strike prices provides completeness. Hahn (1971) showed that, in the presence of transaction costs, completeness was not necessary, and in fact, may be detrimental to efficiency. If transaction costs are allowed for, an infinity of contingent claims markets cannot be supported, and an optimal set must be chosen.

Ross (1976) has shown that a small number of options can provide a very diverse range of payout possibilities. He established the concept of a portfolio of assets called the efficient fund. Options written on such a fund provide the same range of opportunities as options written on the individual assets. Ardetti and John (1980) demonstrated that, when the state space is finite, the probability of an arbitrary portfolio being efficient differs from one by an infinitesimally small number. Nachman (1986) showed that when distributions of asset returns are continuous rather than discrete, no efficient fund exists. Duffie and Shafer (1985) showed that for incomplete markets, equilibrium will exist except on a set of measure zero. Schachter (1986) proved that in the absence of transaction costs, the introduction of an option market at a new strike price cannot reduce welfare in the Pareto sense.

Work has also been done that options may be optimal approximating instruments. Hauser and Eales (1986), among others, suggest Fishburn's (1977) target deviation utility specification as a motivation for the use of options as hedging instruments. While Holthausen (1981) has shown that Fishburn's model satisfies the non-Neumann-Morgenstern expected utility axioms, the model labors under some of its attributes. It holds that marginal utility, below a known critical level of income, bears no connection with marginal utility above this critical level. The reasoning is that there exists critical disaster levels of returns such as bankruptcy or starvation. However, as a disaster looms, economic agents will usually be able to take action to escape. For example, firms will reduce the work

force, salaries or dividends, or enter Chapter 11. Another problem with Fishburn's model relates to the issue of defining the critical value. Associated with any particular production level, there is one critical price. However, if production or other prices are uncertain, there will be a range of critical prices. In addition, even if there are specific circumstances for which options are optimal, there is no guarantee that these circumstances are common.

3. POLYNOMIAL CONTRACTS

A polynomial contract with index i, and expiration date T, is a binding commitment to pay on the expiration date, the price on that date raised to the i^{TH} power, P_T^i.

Polynomial instruments could be written on the underlying price or on futures prices. Their contract specifications would be identical to those of existing options except in the functional form of the payoff. They would have expiration dates and could be traded both long and short.

We see a compelling connection between the econometric approach of flexible functional forms, and the demand for contingent claims. Econometricians have focused on polynomials, in part, because of their well-known approximation properties. A fundamental theorem of real analysis, the Weierstrass theorem (Royden 1988), shows that on a closed interval, the set of polynomial functions can generate a linear combination which converges uniformally to any continuous function. While the set of polynomials is not the only set with this property, polynomials are known to have impressive properties in Banach spaces (Cheney 1966; Acheiser 1956). Essentially their flexibility ensures that they can approximate a large set of functions and accommodate a whole variety of error measurement criteria. Because of the nondifferentiable points that comprise the points of extreme deviation when options are linearly

combined, convex-error measurement criteria will penalize option-based linear approximations.

One may wonder whether polynomials can adequately approximate positions that can be generated with options, but bear in mind that options positions may themselves be only approximations to optimal positions. However, for the moment, consider some typical options positions as being optimal. Figure 1 shows how versatile polynomials are. They can be used to approximate all popular options positions. The long straddle can be approximated using two polynomial contracts, and the vertical bull spread requires three polynomial contracts. In each case, an equivalent number of options markets are used, i.e., the sandwich spread requires four separate options transactions, the purchase of two calls, with a middle strike price and the sale of puts with a low strike price and a high strike price. However, the three strike prices required to create a sandwich spread represent a total of six options markets as puts and calls are traded on each strike price. Thus, the total number of polynomial markets required to replicate these positions is, in all cases, lower than is currently used. In the next sections, we will provide some theoretical situations where polynomials have appeal. We will consider hedging when inputs are flexible, hedging when output is uncertain, and the use of contingent claims to speculate.

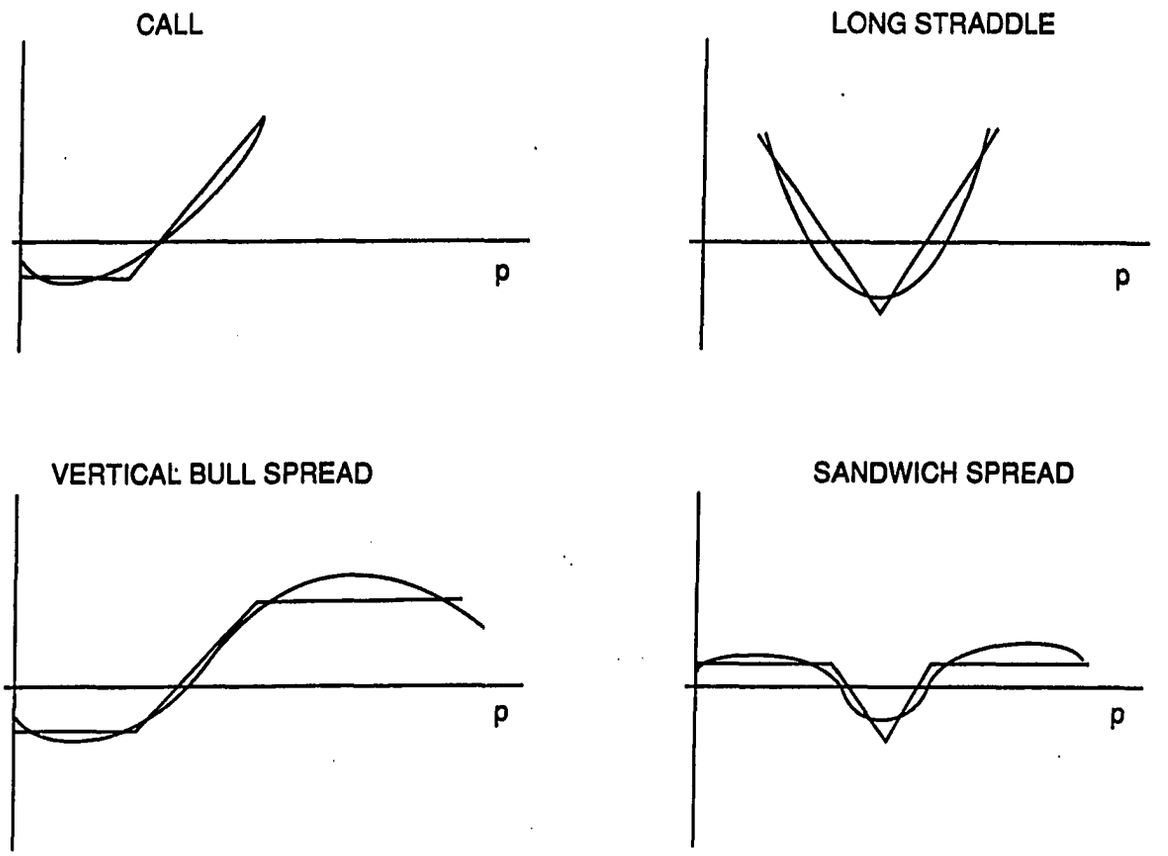

Figure 1. Replicating existing options positions.

4. OPTIMAL INSTRUMENTS FOR HEDGERS

Let $\pi(P)$ be the profit function. Let K(P) be the optimal payoff function, i.e., at a given price realization, P, the desired payoff from financial instruments is the value of K(P). Let f(P) be the market's perception of the price density function. Then the fair market value of any payoff function H(P) is

$$M_H = \int_0^{\infty} H(P)\, f(P)\, dP. \tag{1}$$

Restrict the set of payoff functions to measurable functions G. The investor chooses from G the function K(P) such that

$$E_f U[\pi(P) + K(P) - M_K] \geq E_f U[\pi(P) - M_H] \tag{2}$$

where H(P) is any other measurable function and $E_f U[\cdot]$ is the expectation of utility with respect to the density function $f(\cdot)$. The profit function is convex in price (Chambers 1988).[11] Substitute into (2) above so that the optimal payoff function is that choice of H(P) which maximizes

$$EU[\pi(P) + H(P) - M_H]. \tag{3}$$

This suggests a strong relationship between the optimal payoff function, K(P), and $\pi(P)$.

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[11]Lapan, Moschini, and Hanson have shown that expected utility maximizing hedgers will not use options unless the optimal payoff function is nonlinear in output prices. Possible reasons for this nonlinearity include quasi-fixity of inputs, yield uncertainty, and non-myopic expectations formation.

Proposition: Assume that there are no transaction costs and that financial markets are unbiased. Then for a risk averse producer the optimal payoff function is equal to the negative of the restricted profit function.

Proof: From the concavity of U(·) and from Jensen's inequality

$$U\{\ E[\pi(P)]\ \} \geq E\{\ U[\pi(P) + \xi]\ \} \tag{4}$$

where ξ is a random variable with mean zero. Due to unbiasedness, the difference between any payoff function and its cost will be a random variable satisfying the conditions for ξ. In particular, the expression

$$E\ [\pi(P)] - \pi(P)$$

satisfies the conditions for ξ. Replace ξ by $E[\pi(P)] - \pi(P)$ to find that (4) is satisfied with equality ■.

If K(P) can be synthesized through existing financial instruments, then allocation of resources is efficient in the Arrow-Debreu sense (Arrow 1964). However, to ensure liquidity, an approximation of K(P) may be necessary. The approximation could be, for example, a Taylor's series local approximation (TSLA)

$$K^*(P) = a_0 + a_1P^{\,b} + a_2P^{2b} + a_3P^{3b} \ldots\ldots \tag{5}$$

where when b = 1 it is the conventional Taylor's series expansion. Polynomials,

as do most options positions, tend eventually toward infinity. We may wish to control the rate by specifying that $0 < b < 1$. The coefficient a_o is trivial, and may be ignored. For the TSLA, and assuming the absence of bias, the FOC w.r.t a_1, a_2 and a_3 are

$$E_f\{U'(\pi)[P - E_f(P)]\} = 0 \tag{6a}$$

$$E_f\{U'(\pi)[P^2 - E_f(P^2)]\} = 0 \tag{6b}$$

$$E_f\{U'(\pi)[P^3 - E_f(P^3)]\} = 0, \tag{6c}$$

where $E_f(\cdot)$ denotes expectation with respect to the density function f(P). Solve for a_1, a_2 and a_3 to get $K^*(P)$, the approximation to K(P).

$$K^*(P) = a_1P + a_2P^2 + a_3P^3.$$

It can be seen that the nonlinearity of the restricted profit function leads to optimal payoff functions that are also nonlinear in prices. Options contracts, as they are currently structured, have payouts that are piece-wise linearly related to the maturity price, and may not be as suitable to hedgers as contracts whose value depends on some power of the closing price. The power of polynomials to approximate nonlinear functions is well known. It points to the possibility that one or two carefully chosen polynomial contracts may provide a better approximation to the optimal payoff function than existing options contracts. The choice of these polynomials and the extent to which they might dominate existing options is presented in Section 4.

We next turn to the issue of hedging when output is uncertain. While output is observed only at harvest, futures markets will continually impute a measure of expected output from planting levels, weather data, and other relevant information. Denote by Q_t, the process that the markets impute as output.

Proposition: Let aggregate output, Q_T, be lognormally distributed. Let the absolute price elasticity of demand, ξ, be constant, and let quantity uncertainty be the only source of price uncertainty. Consider a risk averse producer whose individual output, q_T, is uncertain but who knows that the percent change in individual output bears a linear relationship with percentage change in aggregate output[12]

$$q_T = k\, Q_T^{\eta}. \tag{7}$$

If we assume that the basis is always zero, and that k and η are constants, then this producer always desires a nonlinear hedge. The optimal hedging instrument is a contingent claim in a power of the harvest date price, and this power value is $1 - \xi\eta$.

Proof: In the absence of basis risk, we know that harvest date futures price, F_T, equals harvest date cash price, P_T. Because quantity uncertainty is the only source of price uncertainty, the futures price at time $t < T$ is a function only of

[12]Losq (1982) and Sakong et al. (1993) consider the optimal hedge with options under this scenario. While they study the best hedge given existing instruments we optimize over market structure.

time to maturity, and the knowledge at time t, of the statistical properties of Q_T. Treating Q_T as a hypothetical security, which diffuses stochastically over the growing season, and using the lognormality assumption, it can be shown that at any time the following partial differential equation holds[13]

$$\frac{\partial F}{\partial t} + 0.5\, Q_t^2 \sigma_Q^2 \frac{\partial^2 F}{\partial Q^2} = 0, \tag{8}$$

where σ_Q^2 is the instantaneous variance of the log of Q_t.

Because the demand elasticity is constant we can develop the boundary value condition

$$F_T = k_1^{1/\xi} Q_T^{-1/\xi} \tag{9}$$

where k_1 is the constant of proportionality entering the demand equation. The other boundary value condition is

$$Lim_{Q_t \to \infty} F_t = 0. \tag{10}$$

Solving (8) subject to (9) and (10) we get

$$F_t = k_1^{1/\xi} Q_t^{-1/\xi} e^{0.5(1+\xi)\sigma_Q^2 (T-t)/\xi^2}. \tag{11}$$

Rearranging we obtain

[13]This p.d.e was developed by Marcus and Modest (1986). See Constantinides (1978) and Fisher (1978) for a discussion on hypothetical securities.

$$Q_t = k_1 F_t^{-\xi} e^{0.5(1+\xi)\sigma_Q^2(T-t)/\xi}. \tag{12}$$

Because Q_t is lognormally distributed, so too is q_t,

$$q_t = kQ_t^{\eta} = k\{k_1 F_t^{-\xi} e^{0.5(1+\xi)\sigma_Q^2(T-t)/\xi}\}^{\eta}. \tag{13}$$

This is because a lognormally distributed variable raised to a power is also lognormally distributed. From (13) we can see that at planting time (t = 0) the stochastic revenue function faced is

$$F_0 q_0 = k\, k_1^{\eta} F_0^{1-\eta\xi}\, e^{0.5(1+\xi)\eta\sigma_Q^2 T/\xi}. \tag{14}$$

Thus, to hedge completely, the producer will sell forward $k\, k_1^{\eta}\, e^{0.5(1+\xi)\eta\sigma_Q^2 T/\xi}$ contracts in a contingent claim on the $1-\eta\xi$ power of the harvest date futures price, i.e., an instrument that pays off $F_T^{1-\eta\xi}$■.

The result derived above can easily be extended to where the relationship between individual and aggregate production is

$$q_T = a_0 + \Sigma_{i=1}^{m} a_i Q_T^{\eta_i} \tag{15}$$

where a_0, a_1,...,η_1, η_2,...are constants. As the absolute value of demand elasticities for agricultural goods tends to be in the range [0.05, 0.5], and η is comparable to

β in the Capital Asset Pricing Model, the expression $(1 - \eta\xi)$ will usually be positive and probably less than one. Thus, under these conditions, producers will demand polynomials with powers in the range (0, 1).

5. OPTIMAL INSTRUMENTS FOR SPECULATORS

Now consider the case of bias in expectations. Let the individual's subjective assessment of the price distribution be described by the density function s(p). To focus on speculation, the hedging motive is removed by specifying the problem in a pure lottery framework. In this case the goal is to choose from the set of all measurable functions, denoted by {G(P)}, the measurable function K(P) that maximizes

$$\int_0^\infty U[G(P) - M_G(P)]s(P)\,dP \tag{16}$$

$$\text{subject to } M_G(P) = \int_0^\infty G(P)f(P)dP$$

where f(P) remains the markets perceived density function. The solution, K(P), is not unique because functions that differ on sets of measure zero also maximize (16). It may be considered unique up to an equivalence set. A closed form solution to K(P) may not be possible.

Consider the problem of speculation when price is normally distributed[14] and where the speculator concurs with the market's assessment of mean, but

[14]Due to asymmetry, it is more difficult to prove this for the lognormal distribution. However, Monte Carlo simulations, provided later in this paper, suggest that an asymmetric polynomial position is an improvement upon existing options contracts.

believes that variance is higher than the market's assessment.[15] Let F_1 and F_2 denote respectively a futures contract and a contract to pay the square of price on a future date on that date. The alternative sets of instruments are $\{F_1, F_2\}$ and $\{F_1$, option with strike price at mean$\}$.

Assume that the market believes that price has distribution f(P) which is $N(\mu,\sigma^2)$, but that the speculator believes that price has distribution s(P) which is $N(\mu,\tau^2)$. As both price distributions are symmetric about μ, the best speculative position will also be symmetric about μ. Thus on each set of instruments there are two restrictions: symmetry and unbiasedness. Consider the polynomial position

$$K(P) = a_0 + a_1P + a_2P^2. \tag{17}$$

Due to unbiasedness on the part of the market

$$E_f[a_0 + a_1P + a_2P^2] = 0. \tag{18}$$

This implies the restriction

$$a_0 = -a_1\mu - a_2\mu^2 - a_2\sigma^2. \tag{19}$$

Due to symmetry

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[15]Lapan, Moschini, and Hanson have shown that risk averse expected utility maximizing individuals facing a symmetric price distribution will use futures to speculate on the first moment and straddles to speculate on the second moment.

$$\partial K(P)/\partial P|_{p=\mu} = a_1 + 2a_2\mu = 0. \tag{20}$$

Thus the position chosen will be of the form

$$K(P) = a_2(P^2 - 2\mu P + \mu^2 - \sigma^2). \tag{21}$$

This position could also be arrived at by generating a set of orthogonal polynomials. For the normal distribution, such polynomials are called Hermite polynomials (Kelly 1967). Now take the expectation with respect to s(P)

$$E_s[K(P)] = a_2(\mu^2 + \tau^2 - 2\mu^2 + \mu^2 - \sigma^2) = a_2(\tau^2 - \sigma^2) \tag{22}$$

The variance of profit is

$$\begin{aligned} Var[Profit] &= E_s[(profit - E_s[Profit])^2] \\ &= a_2^2 E_s[(P^2 - 2\mu P + \mu^2 - \tau^2)^2] = a_2^2[2\tau^4]. \end{aligned} \tag{23}$$

To arrive at this result it should be noted that for the normal distribution

$$E_s[P^3] = \mu^3 + 3\mu\tau^2,\ E_s[P^4] = \mu^4 + 6\mu^2\tau^2 + 3\tau^4.$$

Now consider the position of a symmetric straddle about the mean, i.e. purchase a put and a call both with strike price μ. Using the symmetry of the normal distribution, it can be seen that the market price for the put should equal that for the call. Denote the straddle by S. It's market price, V(S), is

$$V(S) = 2\int_{\mu}^{\infty} [P - \mu]\, f(P)\, dP = 2Prob[P > \mu]\, \{E_f[P|P > \mu] - \mu\}$$
$$= E_f[P|P>\mu] - \mu.$$

The mean of a truncated normal distribution is

$$E_f[P|P > \mu] = \mu + \sigma\, \phi(0)/\{1-\Phi(0)\}, \tag{24}$$

where: σ =standard deviation of the distribution
$\phi(0) = (2\pi)^{-0.5}$ = probability density function of the standard normal distribution function evaluated at 0.
$\Phi(0) = 0.5$ = cumulative density function of the standard normal distribution function evaluated at 0.

Thus,

$$V(S) = 2\sigma/(2\pi)^{0.5}. \tag{25}$$

The expected speculative gain is the number of straddles purchased times the difference between the subjective and market values of a straddle,

$$E_s[Profit] = b_2\{2\int_{\mu}^{\infty}[P - \mu]\, s(P)\, dP - 2\sigma/(2\pi)^{0.5}\} = b_2\{2(\tau - \sigma)/(2\pi)^{0.5}\}. \tag{26}$$

The variance of this gain is

$$Var[Profit] = b_2^2 E_s[(|P - \mu| - 2(\tau - \sigma)/(2\pi)^{0.5})^2]$$
$$= b_2^2 E_s[(P - \mu)^2 - 4|P - \mu|(\tau - \sigma)/(2\pi)^{0.5} + 2(\tau - \sigma)^2/\pi] \tag{27}$$
$$= b_2^2(\pi\tau^2 - 2\tau^2 + 2\sigma^2)/\pi.$$

For each set of instruments, it is desirable that the ratio of mean profit to standard deviation of profit be high. This ratio, rather than the mean variance

ratio, is chosen because it is homogeneous of degree 0 in the money metric. For the set of instruments $\{F_1, F_2\}$ the mean to standard deviation ratio is

$$(\tau^2 - \sigma^2)/[2\tau^4]^{0.5}.$$

For the set of instruments $\{F_1$, option with strike price at mean$\}$ the mean to standard deviation ratio is

$$\{2(\tau - \sigma)/(2\pi)^{0.5}\}(\pi)^{0.5}/(\pi\tau^2 - 2\tau^2 + 2\sigma^2)^{0.5}$$
$$= (2)^{0.5}(\tau - \sigma)/(\pi\tau^2 - 2\tau^2 + 2\sigma^2)^{0.5}.$$

When $\tau = \sigma$, then the ratio is zero because expected profit is zero, but an open position ensures exposure to risk. To compare the two sets of instruments, take a ratio of the ratios, polynomial position ratio to the options position ratio

$$D = (\tau^2 - \sigma^2)(\pi\tau^2 - 2\tau^2 + 2\sigma^2)^{0.5} / \{[2\tau^4]^{0.5} (2)^{0.5}(\tau - \sigma)\}$$
$$= (\tau^2 - \sigma^2)(\pi\tau^2 - 2\tau^2 + 2\sigma^2)^{0.5} / \{2\tau^2 (\tau - \sigma)\}. \qquad (28)$$

This function is homogeneous of degree zero in τ/σ. D is not defined when $\tau = \sigma$. However, when this is the case, we can use L'Hospital's rule. Differentiate with respect to τ above and below and then evaluate at $\tau = \sigma$. We find

$$Lim_{\tau \to \sigma} D = (\pi)^{0.5} = 1.772 > 1.$$

When $\tau \neq \sigma$ the function may be evaluated. Over the set $\{\tau: 0 < \tau < 1.835\ \sigma, \tau \neq \sigma\}$ D is greater than 1. Thus, the polynomial provides a higher level of expected return per unit risk undertaken than options, provided the speculator's subjective estimate of the volatility is less than 1.8 times the markets estimate of volatility.

Again, there is evidence that polynomials could allow today's multiple options markets be replaced with a smaller number of polynomial contracts, simultaneously increasing the usefulness of the markets to speculators. A measure of what the optimal contract would be and the degree to which participants would benefit, is discussed in the next section.

6. MONTE CARLO SIMULATIONS

The theoretical exercises reported on in Sections 2 and 3 can never be completely convincing. This is true because one could always find a particular situation for which existing options markets provide a perfect payoff. It is easy to see that, to formally solve for a socially optimal set of contingent contracts, we must specify a mass density on each of all possible optimal payoff functions, and then specify a social welfare function aggregating the welfare of individuals. Nevertheless, these results indicate that, for at least some hedgers and speculators, suitably constructed polynomials would increase welfare. The purpose of this section is to report on Monte Carlo simulations to determine the extent to which these markets would improve on existing markets. The first set of simulations compares polynomial and existing options for a typical hedger. The second makes the same comparison for a speculator with private information.

Hedger Simulations

The simulations for both hedgers and speculators are based on a similar procedure. First, the economic situation is laid out. Optimal positions are then compared across market structures using certainty equivalent returns. This procedure essentially measures the point at which the indifference curve between risk and return that passes through the optimal position intersects with the y axis. As long as risk-return indifference curves do not cross, orderings of alternative

outcomes found using this procedure are consistent with utility maximization. The procedure is performed for an individual with a CARA utility function with low and high degrees of risk aversion. The procedure is further discussed in Appendix 1. The results presented in Table 1 compare the certainty equivalent returns (CERs) across a range of market structures. The first column considers a "no financial markets" scenario while the second column considers where futures only exist. The third, fourth and fifth columns consider a two market situation. In each case there is a futures market while the second market is, respectively, a contingent claim in $P^{\frac{1}{2}}$, $P^{1\frac{1}{2}}$ and an option at the mode of the price distribution. Note that the latter can alternatively be viewed as a three market situation where there is both a put and a call at the mode. The final two columns compare the CERs for an individual who must trade contracts in blocks of 1,000 and 5,000 bushels (columns 6 and 7), rather than the single bushel contracts that are implicitly assumed in columns 1 through 5. These last two columns are included because, from the perspective of the hedger, a primary benefit of the polynomials is the movement away from the discrete choices associated with strike prices. Therefore, a valid comparison of the benefits gained from changing market structure is the CER change associated with standardized contracts. The tables do not measure the additional benefits that would arise if polynomials permitted a reduction in the number of markets available, and so, through increased liquidity, reduced the bid-ask spread.

The hedger considered is a 1991 Iowa soybean producer. This was the last

year for which published costs and returns were available. A restricted translog profit function was reconstructed using 1991 Iowa soybean production costs and returns.[16] Because output prices vary considerably and because we wish to use a property of the translog restricted profit function that is valid only at the mean, we considered total economic cost per acre rather than actual crop value per acre. Total economic cost per acre is imputed by placing a fair charge for equity capital and unpaid labor. Total economic cost per acre was $243.06 while per acre variable cash crop expenses amounted to $145.62. The translog restricted profit function, for a single output, is of the form.

$$\pi = Exp[a_0 + a_1 \ln(p) + \tfrac{1}{2} a_{11} (\ln(p))^2 + \Sigma_{i=1}^{n} b_i \ln(w_i) + \tfrac{1}{2} \Sigma_{i=1}^{n} \Sigma_{j=1}^{n} b_{ij} \ln(w_i)\ln(w_j) + \Sigma_{i=1}^{n} c_i \ln(p)\ln(w_i)] \quad (29)$$

where π = profit and p, w_i are the output price and input prices, respectively.

This may be considered as an approximation that is strictly valid only at the point of expansion. This point is where the log of prices is 0. At this point the share of output in restricted profit is a_1, and the own price output elasticity may be evaluated as

$$\varepsilon_{11} = a_1 - 1 + \frac{a_{11}}{a_1}. \quad (30)$$

If the input prices are known with certainty, and if a_1 and a_{11} are known, then we know the shape of the function which is the optimal contingent claims position.

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[16]1991 Iowa Farm Costs and Returns, Iowa State University, University Extension, Ames, Iowa.

We take a soybean own-price supply elasticity, 0.4, from the literature (George and Kirby 1971), and using the previously stated farm economic cost and variable cash crop expenses we arrive at

$$a_1 = \frac{\$243.06}{(\$243.06 - \$145.62)} = 2.4944581 \tag{31}$$

$$a_{11} = \varepsilon_{11} a_1 + a_1(1 - a_1) = -2.73008. \tag{32}$$

The a_0 coefficient is chosen to scale profit up to a level representing a large soybean producer. We assume that such an operation would generate a gross profit of around \$100,000. An a_0 value of Ln (100,000) generates a gross profit of approximately this magnitude. We assume that this is the profit function faced by a producer who places hedges on April 16, 1991, and lifts them upon contract expiration in November 1991. Integrating the profit function with respect to the density function imputed from contingent claims prices, we find expected profit to be \$102,991.4.

The mean of the lognormal distribution was calibrated so that expected settlement price is equal to the futures price. The variance is imputed from the Black (1976) formula using April 16, 1991 prices, the prime interest rate, and a close to the money call (600 c/bu).

The most striking feature of the results presented in Table 1 is the increase in CER when futures markets are introduced. This is particularly true for the risk averse individual. Once producers have hedged the bulk of their price risk, options and polynomials are used to refine the hedge. This refinement is needed

Table 1. 1991 Certainty equivalent returns for a Spring hedge on Iowa soybean farm. $E[\pi]$ = \$102,991.4, σ = 0.1915, π = 100,000 Exp[2.4944581 Log(P/6.085489) - 0.5 (2.73008)Log(P/6.085489)^2]

Risk Aversion Coeff	No Futures	Futures Only	Futures $P^{0.5}$	Futures $P^{1.5}$	Futures Option	Discrete Futures 1,000 bu	Discrete Futures 5,000bu
0.00003	\$86,675	\$102,952	\$102,953	\$102,954	\$102,952	\$102,951	\$102,937
0.00008	\$69,175	\$98,235	\$102,807	\$102,861	\$101,307	\$98,066	\$97,399

due to nonlinearities in the profit function which cannot be hedged with linear futures contracts. In the absence of nonlinearities, neither options or polynomials would be used (Lapan, Moschini, and Hanson 1991). The increase in CER for hedgers depends on the degree of risk aversion. For low risk aversion, the futures market is sufficient to hedge effectively; but, at higher risk aversion, nonlinear contracts have a place. The polynomial contracts fare well in comparison with the existing options markets in the example. The increases in CER among the various options polynomial contracts is not large when compared to the \$100,000 expected profit. However, when we compare the benefits of polynomials with the increase in CER when we go from futures denominated in 5,000 bu to customized futures contracts (\$97,399 versus \$98,235), the increase in CERs with polynomials is more impressive.

Speculator Simulations

Table 2 replicates the simulations performed in Table 1 for a speculator seeking to take advantage of private information on the moments of the price distribution. All of the speculation results involve positions set on April 16, 1993, and lifted on expiration of the November 1993 contract. The CARA utility

Table 2. Certainty equivalent returns for Spring 1993 speculation with soybean contracts

Risk Aversion Coeff	Information Gap	No Futures	Futures Only	Futures $P^{0.5}$	Futures $P^{1.5}$	Futures Option	Discrete Futures 1,000 bu	Discrete Futures 5,000bu
0.00003	1.25μ	$0	$55,652	$56,013	$56,006	$55,964	$55,652	$55,621
	μ/1.25	$0	$55,524	$56,013	$56,011	$55,957	$55,523	$55,465
	1.25σ	$0	$12	$1,676	$1,408	$1,283	$12	$0
	σ/1.25	$0	$12	$1,888	$1,216	$1,649	$11	$0
0.00008	1.25μ	$0	$20,869	$21,004	$21,002	$20,986	$20,869	$20,848
	μ/1.25	$0	$20,821	$21,004	$21,004	$20,984	$20,821	$20,794
	1.25σ	$0	$4	$628	$528	$481	$0	$0
	σ/1.25	$0	$4	$937	$456	$618	$2	$0

function is again used and the risk aversion coefficients are as in Table 1. Because initial wealth does not matter for the CARA function, the results are presented as changes in CERs. The same seven market structures are considered as in Table 1. For each of the risk aversion coefficients, there are four information divergences. In each case, a 25 percent difference between the market and the individual's opinion is considered. Case 1 is where the

individual's assessment of the mean of the log of price is 1.25 times the market's assessment. Case 2 is where the individual's assessment of the mean of the log of price is 1/1.25 times the market's assessment. Cases 3 and 4 are where the individual's assessment of the volatility of the log of price are 1.25 times the market's assessment and 1/1.25 times the market's assessment, respectively. Note that, whereas in Table 1 financial markets were most useful when risk aversion was high, now the reverse is true. Where there is a difference in the assessment of the mean of logs, a large CER is provided by the futures market. Nonlinear instruments increase the CER somewhat and polynomials perform slightly better. The magnitude of this gain is of the order of moving from a minimum futures contract of 5,000 bu to a zero minimum.

Summarizing Tables 1 and 2, a futures market (i.e., a polynomial market with a power value of 1) would appear to satisfy most of the demand for contingent claims. To the extent that a demand for nonlinear payoffs exists, polynomial instruments, and in particular, low power instruments appear more useful than options.

7. VALUING POLYNOMIAL CONTRACTS

Ignore for the moment interest rates, time value, and risk aversion. Assume that there are unbiased markets. To value P^2 consider the definition of variance;

$$\mu_2 = E[(P - \mu_1)^2] \tag{33}$$

where μ_1 = expected value of price = F_1 and μ_i = i^{th} moment about the mean, i > 1.

Expand μ_2 to get

$$F_2 = E[P^2] = \mu_2 + (\mu_1)^2 = \mu_2 + (F_1)^2 \tag{34}$$

where F_i denotes the i^{TH} moment about zero. Thus the value of the P^2 instrument is the variance added to the square of the futures price.

In general

$$\begin{aligned} E[P^i] = \mu_i + iF_1F_{i-1} - [i(i-1)/2](F_1)^2F_{i-2} \\ + [i(i-1)(i-2)/6](F_1)^3F_{i-3} - \ldots\ldots + (i-1)(-1)^i(F_1)^i \end{aligned} \tag{35}$$

where there are i terms. In summary, we can make two statements

1. From the 1,2,..,n^{TH} moments of the markets' assessment of the future price distribution, the price of the contract that pays P^n at settlement can, in efficient markets, be inferred; and
2. From the 1,2,..,n^{TH} polynomial futures prices in efficient markets, the n^{TH} moment of the markets' assessment of the future price distribution can be inferred.

These statements cannot be made about options because, to value options, the

price distribution must be completely specified. If we wish to accommodate the interest rate, r, time to expiration, T-t, and risk aversion in the polynomial pricing formula, we must also specify the price distribution. If we assume a lognormal distribution, we can dynamically combine the contract with the underlying price to purge the system of uncertainty, as in Black and Scholes (1973).

Let P_t denote the asset price at time t and let P_T be the asset price at termination. We need F_i, the value of this claim at time t. Where convenient, we will express F_i as $F_i(P_t)$ to show it's dependence on both the underlying asset price and time. We denote the instantaneous variance of the log of the underlying asset price as σ_P^2. Continuing in the manner of Black and Scholes (1973), we arrive at their nonstochastic partial differential equation problem

$$\frac{\partial F_i}{\partial t} = r F_i - r P_t \frac{\partial F_i}{\partial P_t} - 0.5 \sigma_P^2 P_t^2 \frac{\partial^2 F_i}{\partial P_t^2}$$

$$F_i(P_T) = P_T^i$$

$$F_i(0) = 0.$$

The solution is

$$F_i(P_t) = P_t^i \, e^{0.5i(i-1)\sigma_P^2(T-t)+(i-1)r(T-t)}. \qquad (36)$$

If the claim is written on a futures contract rather than on the underlying asset, then the relevant partial differential equation problem is that arrived at by Black,

$$\frac{\partial F_i}{\partial t} = r\,F_i - 0.5\,\sigma_P^2\,X_t^2\,\frac{\partial^2 F_i}{\partial X_t^2}$$

$$F_i(X_T) = X_T^i$$
$$F_i(0) = 0,$$

where X_t is the futures contract price at time t. The solution is

$$F_i(X_t) = X_t^i\, e^{0.5i(i-1)\sigma_P^2(T-t)-r(T-t)}. \tag{37}$$

Note that the value of a bond, and the value of an ordinary futures contract, are special cases of (37). Further, (36) or (37) can be used to generate a polynomial base with which we can closely approximate the value of any analytic payoff function.[17]

[17]Many businesses have exposure in two prices, i.e., feeder and fed cattle, or oil and exchange rates. Options instruments are not well suited to hedging any price interactions. A contingent claim on the product of the prices may be warranted. For two prices $P_{a,t}$ and $P_{b,t}$ the value of the cross product, $F_{ab,t}$, is $F_{ab,t} = P_{a,t}\,P_{b,t}\,\text{Exp}[\rho_{ab}\,\sigma_a\,\sigma_b\,(T\text{-}t) + r\,(T\text{-}t)]$ where σ_i and ρ_{ab} are respectively $\{\text{Var}[\text{Ln}(P_{i,t})]\}^{1/2}$ and $\text{Cov}[\text{Ln}(P_{a,t}),\text{Ln}(P_{b,t})]/(\sigma_a\sigma_b)$.

8. SUMMARY AND CONCLUSIONS

Options markets, as they are currently structured, evolved in a somewhat haphazard fashion. This paper asks how they might have looked had they been designed within an optimization framework. We describe an alternative market structure that may be more useful than current options markets to both hedgers and speculators and to the markets themselves. These polynomial markets would be continuous nonlinear functions of the futures price, and would be more closely linked to the higher moments of price than are existing options markets.

Currently market operators are trained to think in terms of moments; yet they must act using instruments (options) that are not directly connected with moments. It is because of the consistency of the polynomial contracts with statistical theory that the valuation formulas for polynomials presented in this paper are so simple. Finally, unlike nondifferentiable options, polynomial contracts lend themselves to mathematical analysis.

It is difficult to anticipate whether market participants would be prepared to undergo the learning process required to fully understand and trade polynomial contracts. Existing options markets often require an ability to manipulate position diagrams, calculate fair option values, and understand a relatively large vocabulary; yet, they have been extremely successful. Contracts traded on the square or square root of the underlying asset price, although not as familiar as options, require no additional aptitude for math.

If markets began trading in polynomial contracts, the number of markets

required per underlying asset would fall by at least 50 percent. This is true because the current structure requires trading in both puts and calls, whereas an individual could choose to go long or short on a single polynomial. Because of the direct link between polynomials and the underlying asset price, no separate markets would be required for each of several strike prices. Participants who expected large price moves could purchase a suitable polynomial, and if proven correct, could quite easily liquidate the position. This contrasts with existing options markets, where deep out of the money contracts are either not offered or very thinly traded. Also, investors can often find it difficult to sell deep in the money options at close to their theoretical values.

It is difficult to determine how many polynomials per asset would be supported by the market. The number would be determined by market liquidity, much as the number of strike prices are determined today, but the total number required for each underlying asset, would be less than the number of options markets. Essentially the marketplace would decide the order of a Taylor's series expansion that is economically warranted. In simulations reported here, it appears that almost all the benefit of additional polynomials falls after two or three have been introduced. Thus it is possible that three polynomials could replace the four to five calls and four to five puts that typically trade today. Any reduction in the number of prices quoted per underlying asset would concentrate

market liquidity and consequently reduce bid ask spreads. By offering markets that provide more flexibility and lower transaction costs, the exchanges should benefit from an increase in market volume.

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APPENDIX I

Consider the CARA utility function U(Y(P)] where Y, income, depends on price, P.

$$U[Y(P)] = 1 - e^{-\lambda Y(P)}$$

where λ is the risk aversion coefficient. Income will be the sum of business and financial profits. Using GAUSS software, we numerically integrate utility with respect to the price density function

$$\int_0^{\infty} (1 - e^{-\lambda Y(P)})\, f(P)\, dP = EU.$$

Denote the certainty equivalent return by c. It is the certain income that generates the same utility as a risky income distribution.

$$1 - Exp[-\lambda c] = EU$$

$$\therefore c = \frac{Ln[1 - EU]}{(-\lambda)}$$

It is this value, expressed in $, that is reported in the tables.

PAPER IV

AN ALTERNATIVE PERSPECTIVE ON THE EXPECTED VALUE OF A FUNCTION: ECONOMIC APPLICATIONS

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AN ALTERNATIVE PERSPECTIVE ON THE EXPECTED VALUE OF A FUNCTION: ECONOMIC APPLICATIONS

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ABSTRACT

By approximating the expected value of a function, nonlinear in a stochastic variable, as the sum of values of a sequence of options, we gain additional insights about economic behavior under uncertainty. This is because the respecified behavioral equations contain probabilities and conditional expectations that respond in a predictable manner to changes in the probability distribution. The procedure is formally developed in the context of expected utility maximization when output price is stochastic. It is applied to three problems: to value a risky investment, to study production under price uncertainty, and to study the effect of price uncertainty on expected output when output can be modified in response to realized price.

1. INTRODUCTION

The analysis of decision making under uncertainty deals primarily with the maximization of the expected value of a welfare function. This paper presents an alternative approach to analyzing this class of problem. To understand the marginal effect of an increase in uncertainty it has, in the past, been necessary to place restrictions on either the objective function, or the probability density function (p.d.f). It has also been necessary to be explicit about the meaning of a marginal increase in uncertainty, because different conceptualizations may not have equivalent economic implications [18]. Finally, the technologic environment must be well defined, because both the technology and the decision sequence may alter the effects of uncertainty [10, 30]. Economic results have, inevitably, been on a case-by-case basis. Knowledge has been improved through new perspectives [27, 29], more complicated analytics [14], or the analysis of new situations [10, 30]. The intent of this paper is to present an alternative approach to the maximization problem. It is shown that particular techniques, widely used in finance and statistics, may be applied to problems in the economics of uncertainty.

The earlier work on the effects of uncertainty on economic choices when nonlinearities exist were mainly graphical and intuitive in orientation [22, 28]. By introducing the concept of a mean preserving spread (m.p.s) Rothschild and Stiglitz [25] and Diamond and Stiglitz [7] provided a more analytic theoretic foundation to research in the area. Using this approach, Lippman and McCall

[17] imposed structure on the product of income and the first derivative of the utility function to sign the effect of price uncertainty on the choice variable. Sandmo [27] restricted the concept of an m.p.s to facilitate a straight calculus approach. Much subsequent work has been approached from this perspective [9]. Uncertainty has been widely modeled as a global change from certainty to uncertainty [10, 29], where Taylor series have proved useful, or as a very structured marginal change in uncertainty [3, 9, 21, 30]. In particular, mean-variance analysis has been used extensively in economic models of production [24]. The most general and insightful approach has been through stochastic dominance techniques [5, 21, 23].

In seeking to attribute meaning to the interaction between the objective function and the p.d.f, this paper departs from the traditional approach to optimization under uncertainty. In this paper, we show a link between the second derivative of the objective function and the cumulative density function (c.d.f). To show that the technique has promise, we apply it to three of the most widely studied problems in the uncertainty literature. First, to develop the intuition, we use the methodology to place a bound on the value of a risky investment. The second application, the response of production to a marginal increase in uncertainty, strengthens existing results. The final problem addressed is the effect of a change in the structure of the profit function on expected production when price is uncertain.

The main body of the paper has four sections. First, we develop the

technique in a very general context. We finish the section with an identity equating the conventional expression of expected welfare with an alternative. Analogues of this alternative expression are used to generate results in sections 3, 4, and 5. Section 3 deals with valuing a risky investment. Sections 4 and 5 deal with production under uncertainty and expected profit under uncertainty, respectively. Finally, the paper is summarized and conclusions are drawn.

2. THE MODEL

For the purpose of illustration, we will develop the model in the context of maximizing expected utility. Consider a continuous, piecewise linear approximation to a three-times differentiable concave utility function. Because utility specification is unique up to a positive linear transformation, we can normalize. At the lowest possible income level (price is zero), we can set utility, U, equal to income, Y, and set marginal utility equal to 1. For simplicity, let the approximation consist of three segments. Let A_1 and A_2 be the income levels where marginal utility changes. The approximation is illustrated in Figure 1.

$$(1) \qquad U \approx \begin{cases} Y & \text{for } Y \le A_1 \\ A_1 + a_1(Y - A_1) & \text{for } A_1 < Y \le A_2 \\ A_1 + a_1(A_2 - A_1) + a_2(Y - A_2) & \text{for } A_2 < Y. \end{cases}$$

Let

$$(2) \qquad Y = PQ - C(Q),$$

where Y is income, P is stochastic price at the marketing date, Q is output, and C(Q) is the cost function. Let P_1 and P_2 be the prices where marginal utility falls from 1 to a_1, and from a_1 to a_2 respectively.

$$(3) \qquad \therefore P_1 = \frac{A_1 + C(Q)}{Q},$$

$$(4) \qquad P_2 = \frac{A_2 + C(Q)}{Q}.$$

Denote the probability distribution function of price by f(P). The problem is to

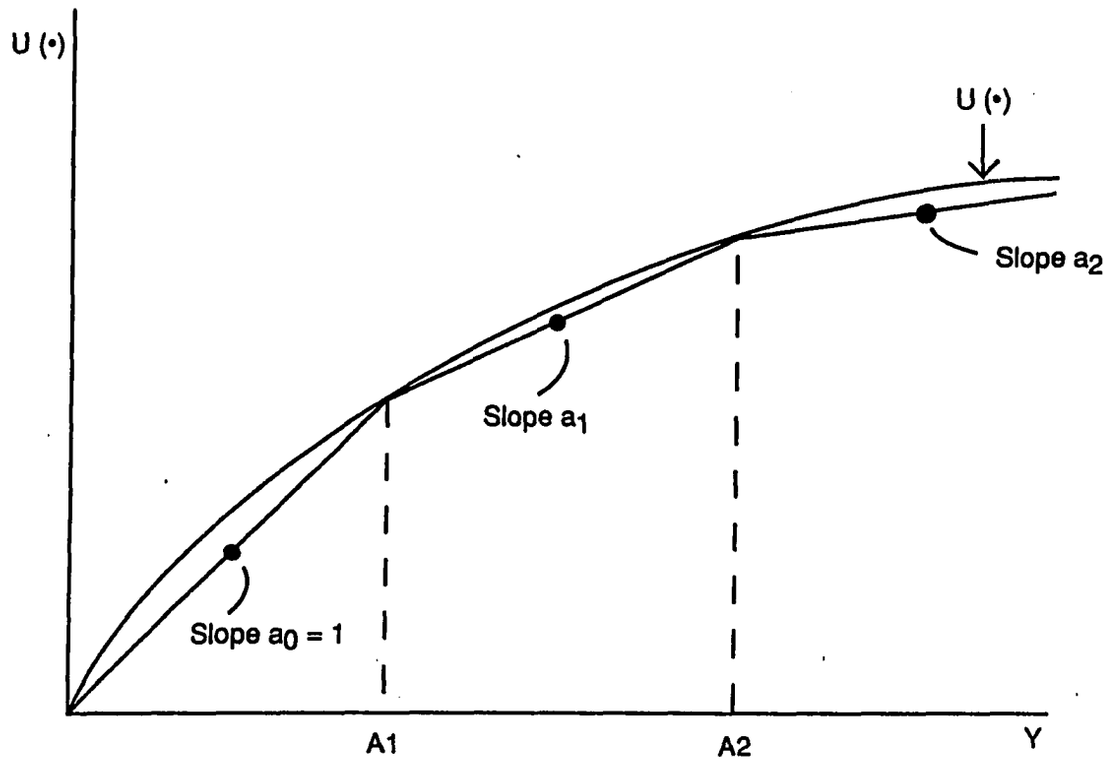

Figure 1. Approximation of utility function.

maximize

$$E[U(Y)] \approx \int_0^{P_1} [PQ - C(Q)]\, f(P)\, dP + \int_{P_1}^{P_2} [A_1 + a_1[PQ - C(Q) - A_1]]\, f(P)\, dP$$
$$+ \int_{P_2}^{\infty} [A_1 + a_1(A_2 - A_1) + a_2[PQ - C(Q) - A_2]]\, f(P)\, dP. \tag{5}$$

Consider Figure 2 where three lines; B1, B2, and B3 are drawn. If these functions are added, we get the supporting, continuous, piecewise linear function in Figure 1. B1 is linear in Y with slope 1. B2 is zero for all Y lower than A_1, and has slope a_1 - 1 after that. Observe that in the financial markets literature, this is a written call[18]. It shows the payoff from a_1 - 1 calls written on Y, where the strike price is A_1. B3 may be interpreted as a_2 - a_1 calls written on Y, where strike price is A_2. If we rearrange (5), we get

$$E[U(Y)] \approx E[P]Q - C(Q) - \{1 - a_1\}Prob(P>P_1)\{E[Y|P>P_1] - A_1\}$$
$$- \{a_1 - a_2\}Prob(P>P_2)\{E[Y|P>P_2] - A_2\}. \tag{6}$$

It can be seen that (6) is expected income plus a weighted sum of expected losses from written calls. Now, refine the approximation so that there are n piecewise continuous linear segments. Reformulate the problem so that $a_1 = 1$ always, and that A_i and a_i do not have the same numerical values as in equation (5). P_i is the

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[18] A purchased call is a contract which grants the purchaser the right, but not the obligation, to purchase a specified asset at specified future date for a specified price (strike price). The option will be exercised only if the asset price on the specified date exceeds the strike price [18]. A written call is the obligation to sell the asset if the call purchaser wishes to buy.

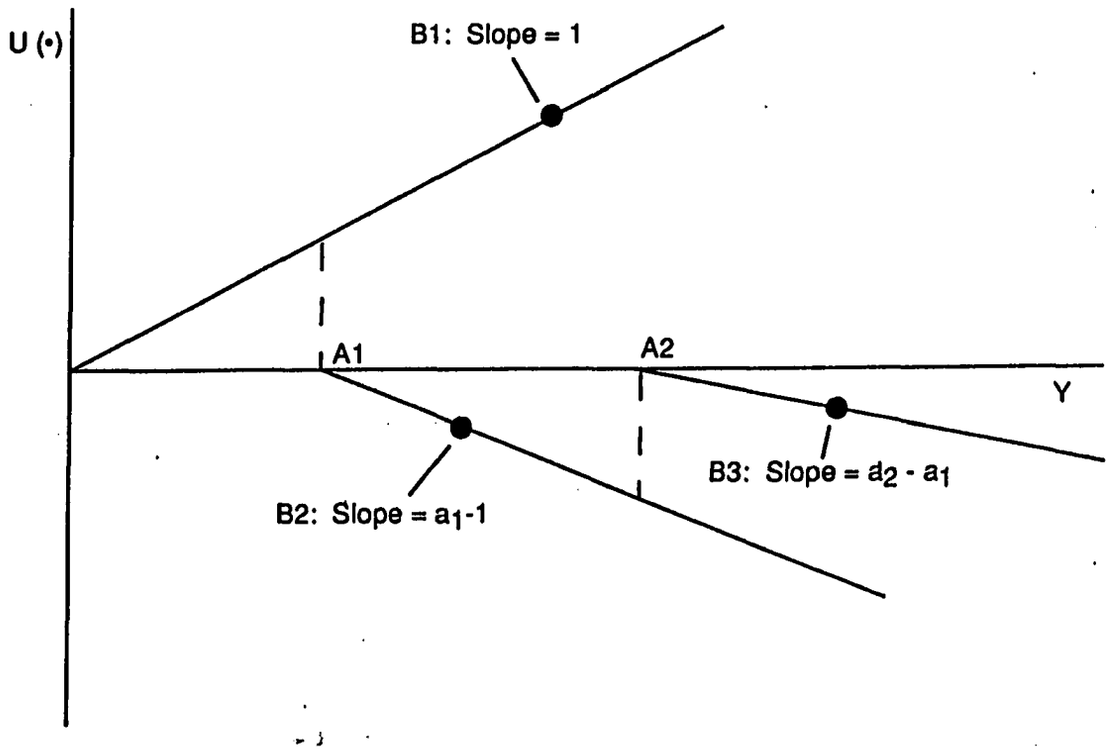

Figure 2. Position diagram decomposition of approximation to utility function.

critical price at which marginal utility changes from a_{i-1} to a_i. From equations (3) and (4), it can be seen that P_i is a function of Q, and may be written as $P_i(Q)$ to emphasize this relationship. Y_i is the income at that critical price. The approximation may be written as

$$E[U(Y)] \approx E[P]Q - C(Q) - \sum_{i=1}^{n} \int_{P_i}^{\infty} (Y - Y_i)\, dP\{a_{i-1} - a_i\}$$

(7)

$$= E[Y] - \sum_{i=1}^{n} \int_{P_i}^{\infty} (Y - Y_i)\, f(P) dP\{a_{i-1} - a_i\}.$$

Note that $a_{i-1} - a_i$, when divided by $A_{i-1} - A_i$, is a discrete approximation to $U''(Y)$. It is necessary, before going further, to summarize some concepts in real analysis. We wish to show that an integration may be substituted for the summation sign in equation (7). In Appendix I, we show that $\int_{P_i}^{\infty} (Y - Y_i)\, f(P)\, dP$ is Stieltjes integrable with respect to a_i, and that, at the limit, we may substitute $U''(Y_i)\, dY_i$ for $a_i - a_{i-1}$. The implication is that, at the limit, equation (7) becomes

$$E[U(Y)] = E[Y] + \int_{Y_{min}}^{\infty} U''(Y_i) \int_{P_i}^{\infty} (Y - Y_i)\, f(P)\, dP\, dY_i. \tag{8}$$

Noting that $Y_i = P_iQ - C(Q)$, we may write (8) as

$$E[U(Y)] = E[Y] + Q \int_{Y_{min}}^{\infty} U''(Y_i) \int_{P_i}^{\infty} (P - P_i)\, f(P)\, dP\, dY_i. \tag{9}$$

We note that the inner integral $\int_{P_i}^{\infty} (P - P_i)\, f(P)\, dP$ is the value formula for a

call option with strike price P_i. Equation (9) may be viewed as comprising two parts; expected income, less the limit of a weighted summation of call options. The second part, in turn, is comprised of two components. One component is due solely to the curvature of the objective function, and does not arise from uncertainty. The other component is due to the interaction of the curvature of the objective function and the price uncertainty. This interaction dependence would be eliminated if price were nonstochastic.

An alternative interpretation of (8) and (9) is to view the function $1 - U'$ as a probability measure. Its value at Y_{min} is 0 by construction. If the utility function is concave, then $1 - U'$ is monotonic increasing, while if we assume that marginal utility at infinite income is zero, then the function has a maximum value of one. The requirement that marginal utility converges to zero is often imposed to avoid the Menger's super St. Petersburg paradox [10]. From this perspective, the second term on the right in equation (8) can be viewed as the negative of the first moment of $Q\int_{P_i}^{\infty}(P - P_i)\, f(P)\, dP$ with respect to the measure $1 - U'$. Using equation (9), or its analogues, we now proceed to our three applications.

3. THE VALUE OF A RISKY INVESTMENT

The problem here does not involve an optimization, but rather the quantitative measurement of the value of a given investment. In this section, the economic agent faces no choices; we seek only to understand the effect of uncertainty on equation (9). First we note that, because $U'' < 0$, any factor that influences call prices uniformly, regardless of strike price, will have a determinate effect on expected utility. Merton [19] has shown that a mean preserving spread (m.p.s) cannot decrease the expected value of a call option, and will increase the value of a call at some strike price. Thus, any m.p.s reduces the expected utility. This conclusion concurs with a proof by Rothschild and Stiglitz [25].

We will now, by imposing the restriction of increasing failure rate (i.f.r) on the p.d.f, develop a lower bound on the value of investment. A distribution is globally i.f.r if, and only if, for each price, the density function divided by one minus the c.d.f is increasing in P for all P. The exponential distribution is globally constant failure rate, i.e $f(P)/[1 - F(P)]$ is independent of P. The normal distribution is i.f.r while the Gamma and Weibull distributions will be either globally i.f.r or nonmonotonic [2]. The lognormal distribution has a nonmonotonic failure rate. We now present a lemma from Henin and Ryan [11]:

Lemma 1: *If the distribution of change in price between now and date of sale is independent of present price, and if the distribution is i.f.r, then the log of the call value is concave in present price.*

The present price provides information concerning the distribution of price at date of sale. Let us respecify $\int_{P_i}^{\infty}(P - P_i)\, f(P)\, dP$ as $\int_{P_i-P_t}^{\infty}(\epsilon + P_t - P_i)\, g(\epsilon)\, d\epsilon$. Here P_t is the present price, ϵ is the price jump between now and date of sale, and $g(\epsilon) = f(P)$. Denote the value of a call with present price P_t and strike price P_i as $W(P_t,P_i)$. If we differentiate $W(P_t,P_i)$ with respect to present price we get

$$\frac{dW}{dP_t} = \int_{P_i-P_t}^{\infty} g(\epsilon)\, d\epsilon = 1 - G(P_i - P_t) = 1 - F(P_i),$$

where G(.) is the c.d.f of ϵ. A second differentiation gives

$$\frac{d^2W}{dP_t^2} = g(P_i - P_t) = f(P_i).$$

The concavity of the log of the call value in present price implies

$$W(P_t,P_i)\left(\frac{d^2W}{dP_t^2}\right) - \left(\frac{dW}{dP_t}\right)^2 < 0,$$

or

$$0 < W(P_t,P_i) < \frac{(dW/dP_t)^2}{d^2W/dP_t^2} = \frac{[1 - F(P_i)]^2}{f(P_i)}.$$

Now we can state

Lemma 2: *If the distribution of change in price between now and date of sale is independent of present price, and if the distribution is i.f.r, then a lower bound on expected utility is* $E[Y] + Q^2E[U''(Y)(H(P))^{-2}]$ *where* $H(P)$ *is* $f(P)/[1 - F(P)]$.

Proof: From equation (9)

$$E[U(Y)] = E[Y] + Q\int_{Y_{min}}^{\infty} U''(Y_i)[\int_{P_i}^{\infty}(P - P_i)\, f(P)\, dP]\, dY_i$$

$$= E[Y] + Q\int_{Y_{min}}^{\infty} U''(Y_i)\, W(P_r, P_i)\, dY_i$$

$$> E[Y] + Q\int_{Y_{min}}^{\infty} U''(Y_i)\, \{[1 - F(P_i)]^2/f(P_i)\}\, dY_i$$

$$= E[Y] + Q^2\int_{0}^{\infty} U''(Y_i)\, [H(P_i)]^{-2}\, f(P_i)\, dP_i$$

$$= E[Y] + Q^2E[U''(Y)(H(P))^{-2}] \qquad \text{Q.E.D.}$$

We can further note that

$$E[U(Y)] > E[Y] + Q^2E[U''(Y)]E[(H(P))^{-2}] + Q^2Cov[U''(Y),(H(P))^{-2}],$$

but $dH^{-2}/dP = -2\,H^{-3}(dH/dP) < 0$ because the density function is i.f.r. Thus, the covariance is positive (negative) according as $U'''(Y) < (>)\ 0$. We can say that, holding $E[U''(Y)]$ constant, the lower bound rises (falls) according as

$U'''(Y) < (>) 0.$

In summary, in this section using the assumption of i.f.r, we developed a lower bound on expected utility. In the next section, we will model uncertainty when there is a decision variable.

4. PRODUCTION UNDER PRICE UNCERTAINTY

Sandmo [27] has shown that a risk-averse decision maker produces less under price uncertainty than under price certainty when the production decision cannot be modified in the light of subsequent knowledge. Ishii [14] proved that nonincreasing absolute risk aversion (non-IARA) is a sufficient condition for quantity to fall under a Sandmo type, marginal increase in price uncertainty. Turnovsky [29] has shown that, when production can be altered later, a risk averse decision maker may plan to produce either more or less than his/her risk neutral counterpart. Some recent work in this area has imposed restrictions on the form of the production function [16]. An alternative approach has been to impose restrictions on the structure of utility [4, 12, 16]. Yet other papers have placed different restrictions on the nature of the mean preserving spread (m.p.s) occurring [8]. Meyer and Ormiston [20] applied stochastic dominance to show that there exist no economic situations for risk averse agents where the choice variable changes uniformly in direction for all mean preserving spreads. They also show that a m.p.s such that all shifted density is relocated to outside the support of the original density function will always reduce output. Davis [6] provided a good analysis of why the production effects of a m.p.s have proven so problematic.

This paper returns to the issue addressed by Sandmo and by Ishii: what is the most general condition under which a risk-averter's production falls as price uncertainty (in the form of a m.p.s) increases? Risk-neutral producers ($U'' = 0$)

will, in Sandmo's model, produce the same before and after a m.p.s in price. In this section, we will show that $U'' < 0$, and a mild condition on the nature of the m.p.s are sufficient to ensure that output falls with a m.p.s in price.

In contrast to the previous section, Q is no longer fixed but may be chosen ex ante. As some of the material in this section involves tedious mathematics, we abbreviate the steps here and include expanded versions in the appendices. Let us note that $Y_{min} = -C(Q)$, and that $P_i = [Y_i + C(Q)]/Q$. Differentiate equation (8) above using Leibnitz rule on the variable bounds of integration (see Appendix II). After some manipulation, we arrive at

$$(10) \qquad \frac{dE[U(Y)]}{dQ} = 0 = E[P] - C' + \int_{Y_{min}}^{\infty} U''(Y_i) \int_{P_i}^{\infty} (P - C')\, f(P)\, dP\, dY_i + C'U''(Y_{min})QE[P],$$

where we let C' denote marginal cost, and $U''(Y_{min})$ be $U''(Y_i)$ evaluated at Y_{min}. Sandmo has shown that under uncertainty, $E[P] > C'$. The term $C'U''(Y_{min})QE[P]$ is negative. The expression $\int_{P_i}^{\infty} (P - C')\, f(P)\, dP$ is always positive, so the term $\int_{Y_{min}}^{\infty} U''(Y_i) \int_{P_i}^{\infty} (P - C')\, f(P)\, dP\, dY_i$ is always negative. Now assume that Q satisfies the first order condition. If a m.p.s occurs, it affects only the expression

$$(11) \qquad B = \int_{Y_{min}}^{\infty} U''(Y_i) \int_{P_i}^{\infty} (P - C')\, f(P)\, dP\, dY_i.$$

As in Sandmo let us assume the second order condition,

$$(12) \qquad \frac{d^2E[U(Y)]}{dQ^2} < 0.$$

Before going further, let us consider the most basic m.p.s from which all other mean preserving spreads can be constructed [25]. See Figure 3 below. Two chunks are removed from the mass, and two are added. The one removed at the lower price is added at an even lower price, while the chunk removed at the higher price is added at an even higher price. This is done so that expected price does not change. Loosely denote the four areas where density changes by P_1 as the highest, then P_2, P_3, and P_4, the lowest. We call the density change around P_j the j^{TH} part of the density change. It can be shown that if one, two, or three parts of a m.p.s occur at a price less than C', then for all values of P_i the expression $\int_{P_i}^{\infty} (P - C') f(P)\, dP$ rises with the m.p.s (see Appendix III). Using the Rothschild and Stiglitz (R&S) step function concept of a m.p.s, we can build a more general distribution that is a m.p.s and reduces output. All distributions whose difference from the original distribution can be decomposed into R&S m.p.s functions such that for each function one, two, or three parts occur at prices less than C' will increase the value of $\int_{P_i}^{\infty} (P - C') f(P)\, dP$ regardless of the value of P_i. One such distribution is illustrated in Figure 4 below. This distribution is built up from m.p.s functions where two parts of the spread occur at either side of marginal cost. In this case there can only be one crossing of the c.d.fs. However, when we build up a m.p.s from combinations of all three cases, then three, or any odd number of crossings can occur. To see this let three parts of

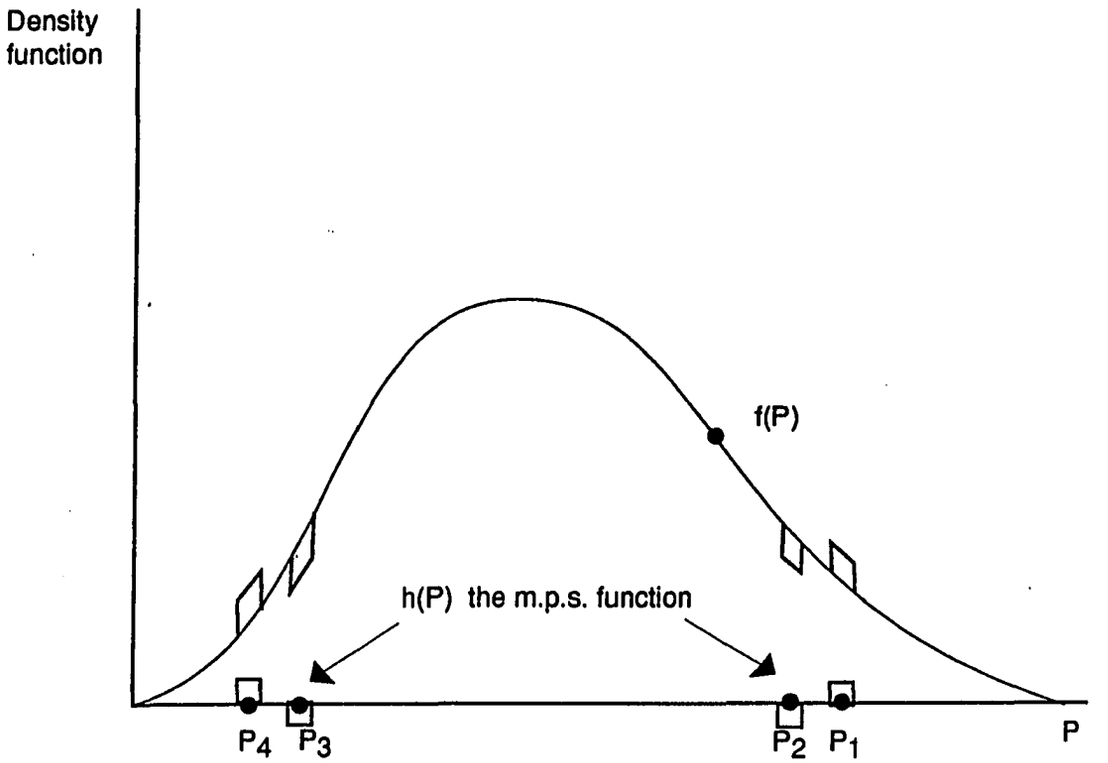

Figure 3. Basic step function mean preserving spread.

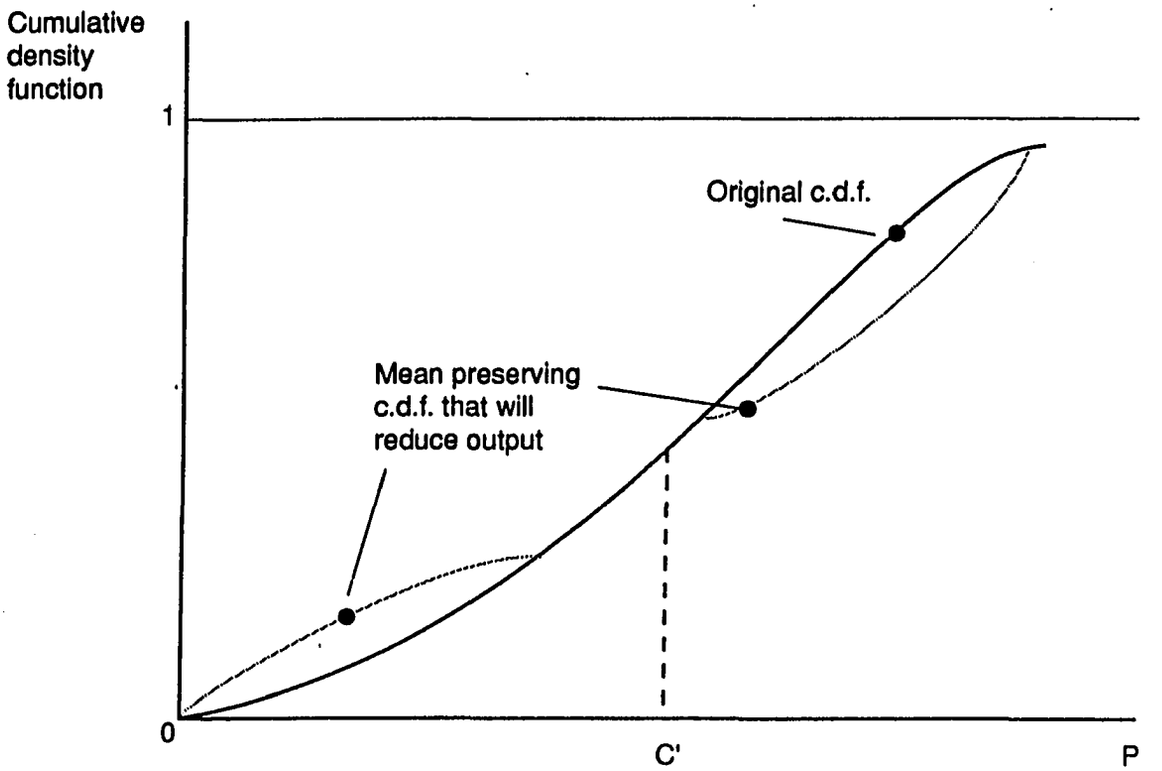

Figure 4. Cumulative density function of an output reducing, mean preserving spread.

one simple m.p.s, call it A, occur below marginal cost, and let one part of another m.p.s, called B, occur below C'. The situation is described in Figure 5 below. Let the one part of m.p.s B below C' be closer to C' than the three parts of A, and let the size of the step, α_B, be large relative to the size of the steps in A. Having adjusted for these two simple m.p.s functions, the new c.d.f will first rise above the old one, return to it, fall below it, and then rise above it again before price exceeds marginal cost. As the part of the m.p.s that is at the highest price, whether it is from A or B, is always positive in direction of displacement, the new c.d.f will always rejoin the old one from below. Thus an odd number of crossings of the c.d.fs must occur for any mean preserving density shift. For the m.p.s shifts A and B considered here, the second and third crossings occur because β_1 can be arbitrarily large.

We are now in a position to state the central result of this section.

Theorem 1: *If* $U''(Y) < 0$, *the second order condition holds, and a m.p.s can be decomposed into R&S m.p.s functions where for each function one, two, or three parts are below* C', *then quantity falls with the m.p.s.*

Proof: We have shown that under these conditions, the value of $\int_{P_i}^{\infty}(P - C')\, f(P)\, dP$ increases for all P_i. Thus, $B = \int_{Y_{min}} U''(Y_i) \int_{P_i}^{\infty}(P - C')\, f(P)\, dP\, dY_i$ must fall with a m.p.s. But this is the only term in (10) that is affected by the m.p.s. Therefore, the right hand side of (10)

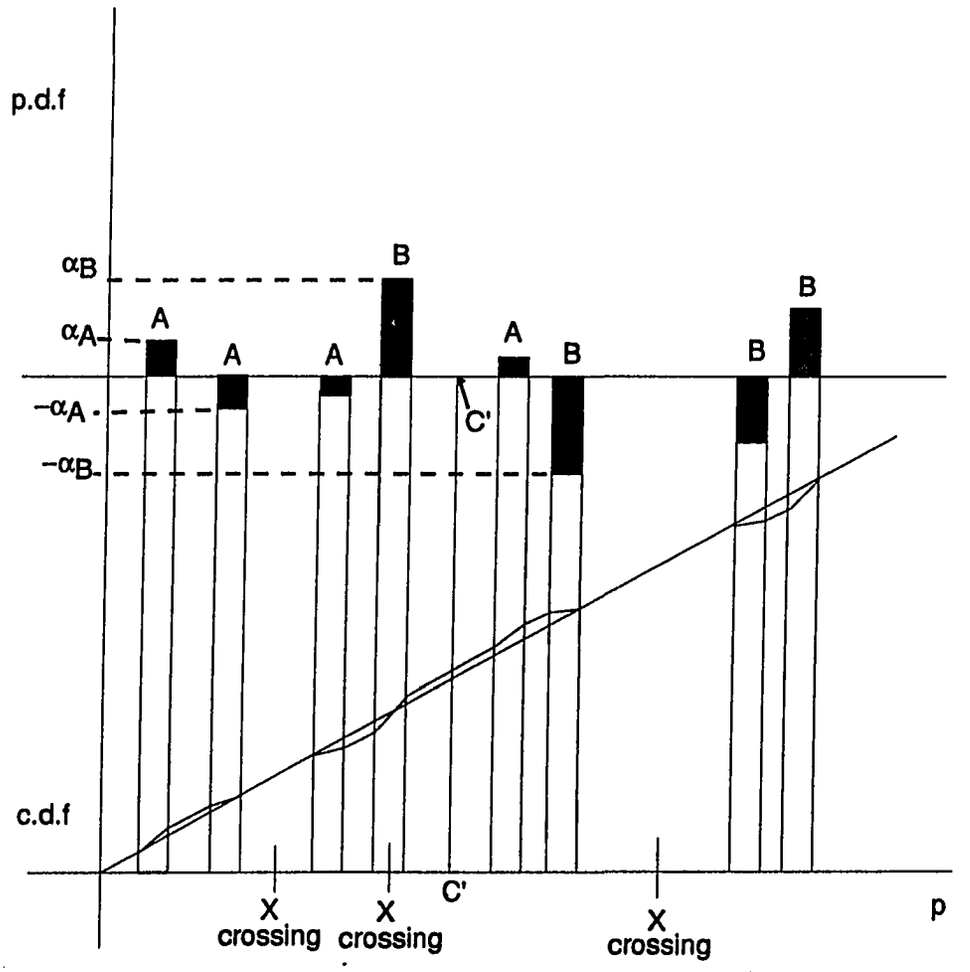

Figure 5. A mean preserving spread that reduces output, and that has a three crossing cumulative density function.

becomes negative and Q must change to restore equilibrium. But from equation (12), the right hand side of (10) falls with an increase in Q. Therefore, Q must fall. Q.E.D.

As this type of m.p.s is a special case of second degree stochastic dominance (SDSD), we cannot infer that SDSD is sufficient to ensure a fall in output. Theorem 1 provides an interesting viewpoint on Sandmo's result that a global increase in risk reduces output. When price is certain, marginal cost equals expected price. In this case a m.p.s necessarily involves moving density from not above C' down to below C', and moving density from not below C' to above it. This situation is covered by theorem 1.

We will now show that output may rise under risk aversion when both the expected price falls and the price distribution is more dispersed. We introduce the concept of a mean altering spread. Viewing Figure 3, a mean reducing spread is where more density is removed from around P_3 and added to around P_4, or less density is removed from around P_2 and added to around P_1. A mean increasing spread is where more density is removed from around P_2 and added to around P_1, or less density is removed from around P_3 and added to around P_4. The following, somewhat counter-intuitive, result places an upper bound on the possible strength of results concerning the effect of price uncertainty on production.

Theorem 2: *For any mean reducing (increasing) spread, there exists a concave utility function with* $U'''(Y) > 0$ *such that production rises (falls).*

Proof: See Appendix IV.

Unlike Theorem 1, in Theorem 2 it was not necessary to impose restrictions on the m.p.s. These two theorems are worthy of some comments. The second lemma concerns self-protection. If expected price falls, there is a disincentive to produce, but there may also be an incentive to produce more in order to reduce the probability of a heavily penalized low income. As the distribution arising from a mean reducing spread is always second degree stochastically dominated by the original distribution, it is plain that being dominated in the SDSD sense is not a sufficient condition for output to fall. Another point is that Ishii's sufficient condition on utility structure (non-IARA) for a marginal m.p.s to reduce output relates only to Sandmo's restrictive form of m.p.s. For the more general m.p.s, it is not clear that Ishii's condition is sufficient because no structure is placed on how density is moved about. We can also see from Theorem 1 that any m.p.s that is confined to extreme price levels will reduce output if the second order condition holds. The price squeeze of Eeckhoudt and Hansen [8] and proposition 3 of Meyer and Ormiston [20] are special cases of this phenomenon. Finally we note, as did Meyer and Ormiston, that $U'' < 0$ is not a sufficient condition for production to fall under a m.p.s. Choose a R&S

m.p.s where either no part or four parts of the function are at prices below C'. At the part of the spread where the the value of $\int_{P_t}^{\infty}(P - C')\, f(P)\, dP$ decreases, let the absolute value of U'' be sufficiently large to cause B to rise with a m.p.s. In this case, output will rise with this particular m.p.s and utility curve.

We now apply the technique to a third problem; how output price uncertainty affects the expected output of an expected profit maximizing firm with some flexibility in choosing output after price is realized.

5. OUTPUT UNDER PRODUCTION FLEXIBILITY

There has been much work done on the implications of technologic flexibility when the output price is uncertain. Oi [22] demonstrated that price uncertainty may benefit producers. Tisdell [28] provided the analogous result for input price uncertainty. The issue of choice of equally flexible inputs under output price uncertainty has been addressed by Batra and Ullah [1]. They show that when some, but not all, inputs can be chosen after output price has become more certain, a nonlinear utility function is not necessary to motivate altered decision-making. They also show that the presence of uncertainty does not change the ratios of marginal productivities. Turnovsky [29] found that the ratios do change if inputs have different levels of flexibility. He also shows that, in a model of quasi-fixed inputs, planned production may rise or fall due to output price uncertainty. This is because flexibility encourages extra investment, whereas risk aversion discourages it. Hartman [10] applied Jensen's inequality to show that, if the marginal value of the quasi-fixed input is concave (convex) in output price, then optimum quasi-fixed investment falls (rises) with uncertainty, provided that the marginal product of the quasi-fixed input is an increasing function of output price. Epstein [9] generalized somewhat on Hartman's results. Wright [30] took a more detailed look at the effect of flexibility on variable factor use and factor proportions under output price uncertainty. He showed that input and output price uncertainty were very similar in effect. This section will consider the effects of uncertainty on production for an expected profit maximizer.

In this section, we will first develop an expression for expected profit in terms of options. We will then use the expression to find expected output and properties of expected output. Define a firm's decision making process as follows: first quasi-fixed inputs are chosen, then output price is revealed and, using this price information, variable input choices are made. Consider a profit function where technology is somewhat flexible so that the function is convex in output price. We consider a firm that seeks to maximize expected profit. As before, we approximate the profit function in a piecewise linear manner. To illustrate, assume that there are three segments. Let A_0 be the approximate profit when price is zero. Let b_0, b_1, and b_2 be the marginal responses to price over the first, second, and third segments respectively. The profit levels at which marginal response rates change are A_1 between the first and second segment, and A_2 between the second and third segment. The approximation is then

$$(13)\qquad \pi(P) \approx \begin{cases} A_0 + b_0 P & \text{for } \pi \le A_1 \text{ or } P < P_1 = \dfrac{A_1 - A_0}{b_0} \\ A_1 + b_1(P - P_1) & \text{for } A_1 < \pi \le A_2 \text{ or } P_1 \le P < P_2 = P_1 + \dfrac{A_2 - A_1}{b_1} \\ A_2 + b_2(P - P_2) & \text{for } A_2 < \pi \text{ or } P_2 = P_1 + \dfrac{A_2 - A_1}{b_1} < P. \end{cases}$$

Now the approximate expected profit function may be written as

$$E[\pi(P)] \approx \int_0^{P_1} [A_0 + b_0 P]\, f(P)dP + \int_{P_1}^{P_2} [A_1 + b_1(P - P_1)]\, f(P)dP$$

$$+ \int_{P_2}^{\infty} [A_2 + b_2(P - P_1)]\, f(P)\, dP.$$

$$= A_0 + b_0 E[P] + \int_0^{P_1} [A_1 - A_0 + (b_1 - b_0)P - b_1 P_1] f(P)\, dP$$

(14)

$$+ \int_{P_2}^{\infty} [A_2 - A_1 + (b_2 - b_1)P + b_1 P_1 - b_2 P_2] f(P)\, dP.$$

As before, introduce more segments to refine the approximation, and then take the limit.

$$E[\pi(P)] = A_0 + b_0 E[P]$$

$$+ \int_0^{\infty}\int_{P_i}^{\infty} [\pi'(P_i) + P\pi''(P_i) - P_i\pi''(P_i) - \pi'(P_i)] f(P)\, dP\, dP_i$$

(15)

$$= A_0 + b_0 E[P] + \int_0^{\infty} \pi''(P_i) \int_{P_i}^{\infty} (P - P_i) f(P)\, dP\, dP_i.$$

We see at once from (15), Oi's result that expected profit rises with a m.p.s because the value of a call rises with a m.p.s. We can also place bounds on the value of expected profit, using arguments similar to those presented when we considered the value of a risky investment.

We now postulate a subsidy that does not affect second, and higher moments about the mean. Adding a price subsidy, s, the expected profit becomes

$$(16) \qquad E[\pi(P+s)] = A_0 + b_0 E[P+s] + \int_0^{\infty} \pi''(P_i) \int_{P_i}^{\infty} (P+s-P_i)\, f(P)\, dP\, dP_i.$$

Differentiating with respect to s we get

$$(17) \qquad \frac{dE[\pi(P+s)]}{ds} = b_0 + \int_0^{\infty} \pi''(P_i)\, [1 - F(P_i)]\, dP_i.$$

If there is a support $[D_1, D_2]$ on the density function, then (17) may be rewritten as

$$\frac{dE[\pi(P+s)]}{ds} = \pi'(D_1) + \int_{D_1}^{D_2} \pi''(P_i)[1 - F(P_i)]\, dP_i.$$

This relationship may be viewed as the welfare effect of a subsidy for a risk neutral producer. Alternatively, we may view (17) as the production effect of an increase in the mean holding constant all other moments around the mean. Interchanging the expectation and differentiation operators, this is admissible as neither lower nor upper price bound depend on the subsidy, we see from Hotelling's lemma that expression (17) represents the expected level of output, E[Q]. We are now in a position to state

Lemma 3: *Given a support on density $[D_1, D_2]$, then an upper bound on expected output is* $\pi'(D_1) + E[P] \int_{D_1}^{D_2} \frac{\pi''(P_i)}{P_i}\, dP_i$.

Proof: The Markov inequality for a nonnegative random variable P gives the inequality $P_i[1 - F(P_i)] \leq E[P]$. Substitute this inequality into the support

adjusted version of equation (17). Q.E.D.

For a given technology, represented by the profit function, no price distribution with the same mean can, in expectation, generate output exceeding this bound. We can go further by noting that the Markov inequality gives $P_i^r[1 - F(P_i)] \leq E[P^r]$, for any r > 0. Now, if given knowledge of any of the moments of price we can place an upper bound on expected output. If given knowledge on several moments we can impose a set of inequalities.

We also note that there are couples of profit and density functions for which the response of expected output to a subsidy is the same. If the profit function changes with a change in price distribution such that $\pi''(P)$ equals $kf(P)/[1 - F(P)]$, where k is a positive constant, then E[Q] is constant. To see this substitute into equation (17),

$$\frac{dE[\pi(P+s)]}{ds} = b_0 + \int_0^\infty \pi''(P_i)[1 - F(P_i)]\, dP_i = b_0 + k.$$

This is independent of s. The increased incentive to produce from increased, expected price is completely counter-balanced by the shift in the profit function. Note that there is a dual relationship between the profit function and the price distribution. The profit function dual (in this sense) to the distribution described by the density function f(P) solves the differential equation

$$\pi''(P) = \frac{kf(P)}{[1 - F(P)]}$$

(18) $\pi(0) = A_0$

$\pi'(0) = b_0.$

For example, for the exponential price distribution $F(P) = 1 - e^{-\lambda P}$ we find

$$\frac{kf(P)}{[1 - F(P)]} = k\lambda,$$

and the solution is the quadratic profit function

(19) $$\pi(P) = A_0 + b_0 P + 0.5k\lambda P^2.$$

Let us now look at the effect of a change in the technology on how beneficial a subsidy is. We ask whether production rises when the curvature is concentrated around one price level, or when it is more dispersed. To do this we must define a concept of curvature concentration. Denote $\pi''(P)$ by y. This may be viewed as the inverse of the slope of the supply curve. For convenience let the price p.d.f be strictly positive only on the domain $[D_1, D_2]$. Define $G = \int_{D_1}^{D_2} \pi''(P)\ P\ dP$. Just as a m.p.s alters the location of density while holding E[P] constant, we will alter the location of $\pi''(P)$ while holding $\int_{D_1}^{D_2} \pi''(P)\ P\ dP$ constant. We will call this operation a m.p.s in curvature. From a marginal cost curve perspective, the marginal cost curve first falls relative to it's original level, and then rises to above the old curve before returning to the original. As A_0 and b_0 are held constant, it can be shown that a dispersion of curvature as described above will generate a profit function that dominates the original profit

function over the domain of the m.p.s in curvature, while at other points in the price domain the old and the new profit functions are identical. Further, the first derivatives of the old and new profit functions are identical at D_1 and D_2. We can consider this technology change as a switch from one machine to a more advanced one, or a change in the way inputs are combined. One might expect that if the profit of a firm rises at some prices and never falls, then expected output should rise. The following theorem shows that whether expected output rises or falls depends upon the shape of the price distribution.

Theorem 3: *Expected output will rise (fall) under a m.p.s in curvature according as* $F(P_i)$ *is concave (convex) in the domain of the m.p.s.*

Proof: We use Rothschild and Stiglitz concept of a mean preserving spread. That is, let the change in distribution function be denoted by $r(y)$, and let

$$r(y) = \begin{cases} \alpha \geq 0 & \text{for } a < x < a + w \\ -\alpha \leq 0 & \text{for } a + v < x < a + v + w \\ -\beta \leq 0 & \text{for } b < x < b + w \\ \beta \geq 0 & \text{for } b + e < x < b + e + w \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < a < a + w \leq a + v < a + v + w < b \leq b + w < b + e + w$, and $\beta e = \alpha v$. We may write the change in E[Q] with a m.p.s in curvature as

$$\Delta \frac{dE[\pi(P+s)]}{ds} = \alpha\{\int_{a}^{a+w} [1-F(P_i)]\, dP_i - \int_{a+v}^{a+v+w} [1-F(P_i)]\, dP_i\} \tag{20}$$
$$- \beta\{\int_{b}^{b+w} [1-F(P_i)]\, dP_i - \int_{b+e}^{b+e+w} [1-F(P_i)]\, dP_i\}.$$

We re-write equation (20) as the limit of a Riemann sum

$$
\begin{aligned}
&Lim_{n\to\infty}\left[\frac{\alpha w}{n+1}\right]\left[\sum_{i=0}^{n}\left\{\left[1-F\left(a+\frac{iw}{n}\right)\right]-\left[1-F\left(a+v+\frac{iw}{w}\right)\right]\right\}\right] \\
&-Lim_{n\to\infty}\left[\frac{\beta w}{n+1}\right]\left[\sum_{i=0}^{n}\left\{\left[1-F\left(b+\frac{iw}{w}\right)\right]-\left[1-F\left(b+e+\frac{iw}{n}\right)\right]\right\}\right] \\
&=Lim_{n\to\infty}\left[\frac{\alpha w}{n+1}\right]\left[\sum_{i=0}^{n}\left\{F\left(a+v+\frac{iw}{n}\right)-F\left(a+\frac{iw}{w}\right)\right\}\right] \\
&-Lim_{n\to\infty}\left[\frac{\beta w}{n+1}\right]\left[\sum_{i=0}^{n}\left\{F\left(b+e+\frac{iw}{w}\right)-F\left(b+\frac{iw}{n}\right)\right\}\right].
\end{aligned}
\tag{21}
$$

Now let us invoke the mean value theorem[19]. Replace $F(a+v+iw/n)-F(a+iw/n)$ with $vF'(\xi_i)$, where $F'(\xi_i)$ is the partial derivative with respect to price evaluated at ξ_i, and where $a+iw/n < \xi_i < a+v+iw/n$. Replace $F(b+e+iw/n)-F(b+iw/n)$ with $eF'(\eta_i)$, where $b+iw/n < \eta_i < b+e+iw/n$. Thus equation (21) may be rewritten as

$$
\begin{aligned}
\Delta\,\frac{dE[\pi]}{ds} &= \alpha w\, Lim_{n\to\infty}\left[\frac{1}{n+1}\right]\left[\sum_{i=0}^{n} vF'(\xi_i)\right] \\
&-\beta w\, Lim_{n\to\infty}\left[\frac{1}{n+1}\right]\left[\sum_{i=0}^{n} eF'(\eta_i)\right] \\
&=\alpha vw\, Lim_{n\to\infty}\left[\frac{1}{n+1}\right]\left[\sum_{i=0}^{n}\left\{F(\xi_i)-F(\eta_i)\right\}\right].
\end{aligned}
\tag{22}
$$

---

[19] The mean value theorem states that if $W(P_i)$ is a continuous function on $[a,b]$ and differentiable on (a,b) then, there is a point ξ_i in (a,b) at which $W(a) - W(b) = (b-a)W'(\xi_i)$.

But $\eta_i - \xi_i > b - a - v > 0$. If $F(P_i)$ is strictly concave then $F'(\xi_i) - F'(\eta_i) > 0$. We also know that $F'(\xi_i) - F'(\eta_i) > F'(a+v) - F'(b) > 0$. Let

$$F'(a+v) - F'(b) = \delta,$$

and substitute into equation (22),

$$\Delta \frac{dE[\pi]}{ds} \geq \alpha v w Lim_{n\to\infty}\left[\frac{1}{n+1}\right]\left[\sum_{i=0}^{n} \delta\right] = \alpha v w \delta Lim_{n\to\infty}\left[\frac{n}{n+1}\right] = \alpha v w \delta > 0.$$

For $F(P_i)$ convex $F'(\xi_i) - F'(\eta_i) < F'(a+v) - F'(b) < \delta < 0$ and

$$\Delta \frac{dE[\pi]}{ds} = \Delta E[Q] < 0. \qquad \text{Q.E.D.}$$

Thus, even though the technology change is such that the altered profit function dominates the original profit function at some prices and is never dominated by the original profit function, expected output may fall. Whether it rises or falls depends on the curvature of the price density function in the locality of the change in profit function curvature. For example, if price is normally distributed then a curvature change that preserves G will increase expected output if all the curvature change occurs at prices close to E[P], but will decrease expected output if all the curvature change occurs at either tail of the price distribution. We can go further by applying Jensen's inequality to equation (17). Let the c.d.f be strictly concave (convex) on the domain $[E_1, E_2]$, and define $H = \int_{E_1}^{E_2} \pi''(P)\ P\ dP / [\pi'(E_2) - \pi'(E_1)]$. From theorem 3 and Jensen's inequality, if H is constant then expected output must exceed (cannot exceed)

$$b_0 + \int_0^{E_1} \pi''(P_i)[1-F(P_i)]\,dP_i + [1-F(H)][\pi'(E_2)-\pi'(E_1)]$$

$$+ \int_{E_2}^{\infty} \pi''(P_i)[1-F(P_i)]\,dP_i.$$

The technology, or profit function, that generates this lower (upper) bound on expected output is that constructed by extrapolating the tangent of the profit function at E_1 forward and extrapolating the tangent of the profit function at E_2 backward. The intersection is at H where the profit function is kinked. Thus, the technology that produces the lower (upper) bound on expected output has zero curvature on (E_1, H) and on (H, E_2). The curvature is all concentrated at H where it is infinite and may be represented by the limit of a Dirac delta function, a function used to model impulses in the theory of differential equations.

6. CONCLUSIONS

In this paper we have shown that the existing body of knowledge on financial option theory can improve our understanding of the theory of the firm under uncertainty. The analogy would seem to possess considerable potential. We have applied our technique to three of the most intensively researched problems in the theory of the firm under price uncertainty. In the first application, the value of a risky investment, we show how a lower bound can be placed on the investment value. In the second application, production under uncertainty, we show that when the utility function has a negative second derivative, and when a m.p.s satisfies a condition somewhat stronger than second degree stochastic dominance, then output falls with the m.p.s. This condition permits multiple crossings of the c.d.fs. It is also shown that second degree stochastic dominance cannot be a sufficient condition for output to fall under a m.p.s. We prove that, perhaps counter-intuitively, production may rise if expected price falls and price becomes more dispersed. Finally, we apply the technique to the profit function under uncertainty. We develop an expression for expected output and find that, if we hold a particular function of profit function curvature constant, then a spread of the profit function curvature increases (decreases) expected output if the price c.d.f is concave (convex) in the locality of the curvature change. While each problem can eventually be solved using stochastic dominance techniques, this approach provides insights into the economics of the problem at hand and into the meaning of stochastic dominance. Curvature is seen as generating

options on the stochastic variable.

A particularly interesting aspect of the results presented is the way the c.d.f, rather than moments, are important in determining economic effects. The well-developed statistical theory of reliability, with its emphasis on c.d.fs, may hold fruitful implications for the theory of the firm under uncertainty.

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APPENDIX I

Riemann Integration: Partition an interval [c, d] on the domain of Y into n segments. Thus $c = Y_0 \leq Y_1 \leq \ldots\ldots \leq Y_{n-1} \leq Y_n = d$. A partition, X, is described by the points $\{Y_0, Y_1, \ldots\ldots Y_{n-1}, Y_n\}$. Consider the function g(Y). Define

$$M_i = sup\ g(Y)\ \ such\ that\ \ Y_{i-1} \leq Y \leq Y_i.$$

$$m_i = inf\ g(Y)\ \ such\ that\ \ Y_{i-1} \leq Y \leq Y_i.$$

$$U(X, g) = \sum_{i=1}^{n} M_i\ \Delta Y_i,\ L(X, g) = \sum_{i=1}^{n} m_i\ \Delta Y_i.$$

The U(X, g) and L(X, g) are upper and lower bounds, respectively, on the integration over the partition X. Define

$inf_X U(X, g)$ = *inf of* $U(X, g)$ over all possible partitions = Upper Riemann integral.

$sup_X L(X, g)$ = *sup of* $L(X, g)$ over all possible partitions = Lower Riemann integral.

If $\inf_X U(X, g) = \sup_X L(X, g)$, then the function g(Y) is Riemann integrable over [c, d].

Stieltjes Integration: Let $\alpha(Y)$ be a monotonically increasing function on [c, d]. The function may, as in the case that shall be considered, be monotonically decreasing also. However, to be consistent with how textbooks explain Stieltjes integration, we shall assume that $\alpha(Y)$ is monotonically increasing. Denote

$$\Delta\ \alpha_i = \alpha(Y_i) - \alpha(Y_{i-1}),$$

$$U(X,g,\alpha) = \sum_{i=1}^{n} M_i \Delta \alpha_i, \; L(X,g,\alpha) = \sum_{i=1}^{n} m_i \Delta \alpha_i.$$

If $\inf_X U(X, g, \alpha) = \sup_X L(X, g, \alpha)$, then the function g(.) is Stieltjes integrable over [c, d]. Now refer to Theorem 6.9 in Rudin [26].

Theorem: *If $g(Y_i)$ is monotonic on [c, d] and if $\alpha(Y_i)$ is both monotonic and continuous on [c, d], then $g(Y_i)$ is Stieltjes integrable.*

Now refer back to equation (7). Let $g(Y_i)$ be $\int_{P_i}^{\infty} (Y - Y_i) f(P)\, dP$. Replace $\alpha(Y_i)$ with a_i, the marginal utility at Y_i. We wish to show that (7) is Stieltjes integrable and, at the limit, the summation sign may be replaced by an integration sign. We need to show that $\int_{P_i}^{\infty} (Y - Y_i) f(P)\, dP$ is monotonic in Y_i. Remember that Q is held constant. From (3) and (4) we see that $P_i = [Y_i + C(Q)]/Q$ so

$$\frac{\partial P_i}{\partial Y_i} = Q^{-1}.$$

Derive $\int_{P_i}^{\infty} (Y - Y_i) f(P)\, dP$ with respect to Y_i, noting that P_i is a function of Y_i,

$$\frac{\partial g}{\partial Y_i} = -\int_{P_i}^{\infty} f(P)\, dP - [P_i Q - C(Q) - Y_i] f(P_i) \frac{\partial P_i}{\partial Y_i} = -\int_{P_i}^{\infty} f(P)\, dP.$$

Thus, the function is monotonic. We also need to show that $\alpha(Y_i) = a_i = U'(Y_i)$ is continuous and monotonic. If U'' exists then U' must be differentiable, and so must be continuous. If $U'' < 0$, then $\alpha(Y_i) = a_i = U'(Y_i)$ is monotonic. Thus,

at the limit we can change variables (see Rudin Theorem 6.19) and so replace (7) with

$$E[U(Y)] = E[Y] + \int_{Y_{min}}^{\infty}\left[\int_{P_i}^{\infty}(Y - Y_i)\, f(P)dP\right] dU'.$$

where Y_{min} is income when price is zero. Now replace dU' with $U''dY_i$,

$$\frac{dU'}{dY_i} = U'' \rightarrow dU' = U''\, dY_i.$$

By substitution we arrive at

$$(8) \qquad E[U(Y)] = E[Y] + \int_{Y_{min}}^{\infty} U''(Y_i)\left[\int_{P_i}^{\infty}(Y - Y_i)\, f(P)dP\right] dY_i.$$

APPENDIX II

Differentiate equation (8) above using Liebnitz rule on the variable bounds of integration. Note that $Y_{min} = -C(Q)$, and $P_i = [Y_i + C(Q)]/Q$.

$$\frac{dE[U(Y)]}{dQ} = 0 = E[P] - C' + \int_{Y_{min}}^{\infty} U''(Y_i) \int_{P_i}^{\infty} (P - C') f(P)\, dP\, dY_i$$

$$- \int_{Y_{min}}^{\infty} U''(Y_i)[Y_i - Y_i] f(P_i) \left(\frac{\partial P_i}{\partial Q}\right) dY_i$$

$$- U''(Y_{min}) \int_{0}^{\infty} [PQ - C(Q) - Y_{min}] f(P)\, dP \frac{\partial Y_{min}}{\partial Q}.$$

The $\int_{Y_{min}}^{\infty} U''(Y_i)[Y_i - Y_i]\, f(P_i)(\partial P_i/\partial Q)\, dY_i$ term cancels. As $Y_{min} = -C(Q)$, and $\partial Y_{min}/\partial Q = -C'$, we can write

$$- U''(Y_{min}) \int_{0}^{\infty} [PQ - C(Q) - Y_{min}] f(P)\, dP \frac{\partial Y_{min}}{\partial Q}$$

$$= C' U''(Y_{min}) Q \int_{0}^{\infty} P f(P)\, dP = C' U''(Y_{min}) Q E[P].$$

Therefore, the first order condition is

$$0 = E[P] - C' + \int_{Y_{min}}^{\infty} U''(Y_i) \int_{P_i}^{\infty} (P - C') f(P)\, dP\, dY_i + C' U''(Y_{min}) Q E[P]. \tag{10}$$

APPENDIX III

We refer to Figure 3. There are five possible situations: all the density changes occur above C', three of the changes occur above C', two occur above it, one occurs above it, or all occur below it. Let us now consider the case where three changes occur above it. We will use the notation in Figure 3. Each density change has width 2Δ.

On $[P_1 - \Delta, P_1 + \Delta]$ the value of h(P) is β.

On $[P_2 - \Delta, P_2 + \Delta]$ the value of h(P) is $-\beta$.

On $[P_3 - \Delta, P_3 + \Delta]$ the value of h(P) is $-\alpha$.

On $[P_4 - \Delta, P_4 + \Delta]$ the value of h(P) is α.

The m.p.s ensures that

$$\int_0^\infty h(P)\, dP = 0,$$

and

$$\int_0^\infty P\, h(P)\, dP = 0.$$

When $P_i > P_1$, then $\int_{P_i}^\infty (P - C')\, h(P)\, dP$ does not change.

When $P_1 > P_i > P_2$, then

$$\int_{P_i}^\infty (P - C')\, h(P)\, dP = \int_{P_1-\Delta}^{P_1+\Delta} (P-C')\beta\, dP$$
$$= 2\Delta\beta(P_1 - C') > 0.$$

When $P_2 > P_i > P_3$, then

$$\int_{P_i}^\infty (P - C')\, h(P)\, dP = \int_{P_1-\Delta}^{P_1+\Delta} (P-C')\beta\, dP - \int_{P_2-\Delta}^{P_2+\Delta} (P-C')\beta\, dP$$
$$= 2\Delta\beta(P_1 - P_2) > 0.$$

When $P_3 > P_i > P_4$, then

$$\int_{P_i}^{\infty}(P - C')\, h(P)\, dP = -\int_{P_4-\Delta}^{P_4+\Delta}(P-C')\alpha\, dP = 2\Delta\alpha(C' - P_4) > 0.$$

When $P_4 > P_i$, then

$$\int_{P_i}^{\infty}(P - C')\, h(P)\, dP = \int_{0}^{\infty}(P-C')h(P)\, dP = 0.$$

Thus, in all cases the m.p.s increases the value of the function regardless of the value of P_i. The cases where two changes, and one change occur above marginal cost yield identical qualitative results.

Now we will show that violations may occur when the entirity of the m.p.s occurs either above or below marginal cost. Consider when the m.p.s is all above marginal cost.

When $P_i > P_1$, then $\int_{P_i}^{\infty}(P - C')\, h(P)\, dP$ does not change.

When $P_1 > P_i > P_2$, then

$$\int_{P_i}^{\infty}(P - C')\, h(P)\, dP = \int_{P_1-\Delta}^{P_1+\Delta}(P-C')\beta\, dP = 2\Delta\beta(P_1 - C') > 0.$$

When $P_2 > P_i > P_3$, then

$$\int_{P_i}^{\infty}(P - C')\, h(P)\, dP = \int_{P_1-\Delta}^{P_1+\Delta}(P-C')\beta\, dP - \int_{P_2-\Delta}^{P_2+\Delta}(P-C')\beta\, dP$$
$$= 2\Delta\beta(P_1 - P_2) > 0.$$

When $P_3 > P_i > P_4$, then

$$\int_{P_i}^{\infty}(P - C')\, h(P)\, dP = -\int_{P_4-\Delta}^{P_4+\Delta}(P-C')\alpha\, dP = 2\Delta\alpha(C' - P_4) < 0.$$

When $P_4 > P_i$, then

$$\int_{P_i}^{\infty}(P - C')\, h(P)\, dP = \int_{0}^{\infty}(P-C')h(P)\, dP = 0.$$

Thus, if the absolute value of U'' is very large when $P_3 > P_i > P_4$, and all the

m.p.s is above marginal cost, then the value of

$B = \int_{Y_{min}}^{\infty} U''(Y_i) \int_{P_i}^{\infty} (P - C') f(P)\, dP\, dY_i$ may fall. This, in turn, will lead to an increase in output. Therefore, second degree stochastic dominance, in addition to risk aversion, does not ensure increased output.

Consider when the m.p.s is all below marginal cost.

When $P_i > P_1$, then $\int_{P_i}^{\infty} (P - C')\, h(P)\, dP$ does not change.

When $P_1 > P_i > P_2$, then

$$\int_{P_i}^{\infty} (P - C')\, h(P)\, dP = \int_{P_1-\Delta}^{P_1+\Delta} (P - C')\beta\, dP = 2\Delta\beta(P_1 - C') < 0.$$

When $P_2 > P_i > P_3$, then

$$\int_{P_i}^{\infty} (P - C')\, h(P)\, dP = \int_{P_1-\Delta}^{P_1+\Delta} (P - C')\beta\, dP - \int_{P_2-\Delta}^{P_2+\Delta} (P - C')\beta\, dP$$
$$= 2\Delta\beta(P_1 - P_2) > 0.$$

When $P_3 > P_i > P_4$, then

$$\int_{P_i}^{\infty} (P - C')\, h(P)\, dP = -\int_{P_4-\Delta}^{P_4+\Delta} (P - C')\alpha\, dP = 2\Delta\alpha(C' - P_4) > 0.$$

When $P_4 > P_i$, then

$$\int_{P_i}^{\infty} (P - C')\, h(P)\, dP = \int_{0}^{\infty} (P - C')h(P)\, dP = 0.$$

From Rothschild and Stiglitz [25] theorem 1(b) it follows that

a) any mean preserving change in the density function that can be decomposed into mean preserving spreads, none of which occur either entirely below marginal cost or entirely above marginal cost will reduce output.

b) any mean preserving change in the density function that can be decomposed into mean preserving spreads all of which occur either in $[0, C']$ or $[C', \infty)$, but

can not be decomposed as in a), will have a quantity effect that depends necessarily on the interaction between the utility curve and the c.d.f change.

Of other mean preserving changes in the c.d.f nothing can be said at present. The issue involves set theory. In particular the theory of measures, signed measures, and measure decomposition may provide answers.

APPENDIX IV

Theorem 2: *For any mean reducing (increasing) spread, there exists a concave utility function with* $U'''(Y) > 0$ *such that production rises (falls).*

Proof: We will prove for the mean reducing case. The other case is almost identical. Following the analysis in Appendix III, for P_i below the lowest chunk in h(P), $\int_{P_i}^{\infty} P\, g(P)\, dP < \int_{P_i}^{\infty} P\, f(P)\, dP$, while $\int_{P_i}^{\infty} C'\, g(P)\, dP = \int_{P_i}^{\infty} C'\, f(P)\, dP$. Thus, there is a strictly positive price P^* for which if P_i is in $[0, P^*)$, then the value of $\int_{P_i}^{\infty} (P - C')\, f(P)\, dP$ falls with a mean reducing spread. Let $U''(Y_{min})$ be absolutely very large, and let it rise steeply as P_i rises from 0 to P^*. Such a path for $U''(Y)$ can be chosen so that

$$B = \int_{Y_{min}}^{\infty} U''(Y_i) \int_{P_i}^{\infty} (P - C') f(P)\, dP\, dY_i$$

rises with a mean reducing spread. In (10), $C'U''(Y_{min})QE[P]$ will also rise with a mean reducing spread. While E[P] will fall, it is admissible to choose $U''(Y_{min}) < -1/[QC']$ so that (10) will assuredly rise. From the second order condition, if the right hand side of (12) is positive, then Q must rise to restore equilibrium. Thus Q must rise. Q.E.D.

GENERAL SUMMARY

In this thesis contingent claims techniques have been applied to the problem of optimizing the expected value of a welfare function. In paper I we consider the relationship between financial market completeness, corn production, and the corn target price program. Using the observation that the program is similar to a put option issued by the government, we found that the per acre program benefit in 1993, at around $20/acre, was quite large. We also found that the program encourages producers to engage in the trading of contingent claims, and that the existence of contingent markets facilitates the policy maker's goal of decoupled agricultural support. In paper II we proposed a method for estimating the expected cost to the government of the corn target price program. The model allows the government to understand the implications for output and budget control of different program parameter choices This model may be adapted to other economic problems, such as the effects of minimum wage or rent control laws on production and factor use. In paper III we suggest that there is an inconsistency between the structure of existing contingent claims markets and how economists would seem to prefer to approximate demand functions. We propose an alternative structure that is consistent with the preferred approach to demand function approximation, and with the moment based foundations of statistics. In the final paper we propose an alternative perspective on problems involving the maximization of the expected value of a welfare function. We reformulate the objective function in terms of options. We then show that

existing techniques from economics, statistics, and finance theory may be applied to better understand the economic effects of uncertainty. Three standard economic problems are considered; valuation of a risky investment, production under price uncertainty, and the effects of price uncertainty on expected profit.

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